CAAP Math C, Mr. Church, Study Questions
`http://www.math.uchicago.edu/~tchurch/`

Here are some selected problems and exercises that you might find it useful to study. (The test will of course be shorter than this.) I recommend trying to solve every question if you can. For the questions where "prove your answer is correct" is mentioned separately, I mean it. I also recommend reviewing the definition of a group and of a commutative ring. You should study the axioms, but you don't need to memorize them—they will be provided if necessary. You are all well prepared for this test. Good luck!

1. Compute the greatest common divisor of pairs of numbers, for example $\gcd(7, 12)$, $\gcd(6, 15)$, $\gcd(10, 40)$, $\gcd(99999, 99)$. Try some other pairs.

2. For each pair a, b, find integers x and y so that you can write the greatest common divisor in the form $\gcd(a, b) = ax + by$. For the above examples, this is: find x and y so that $\gcd(7, 12) = 7x + 12y$; find x and y so that $\gcd(6, 15) = 6x + 15y$, and find x and y so that $\gcd(10, 40) = 10x + 40y$.

3. Define $\mathcal{E} = \{n \in \mathbb{Z} | n \text{ is even}\}$ to be the set of even integers. Is $\mathcal{E}$ a group under the operation of addition?

4. Let the function $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x - 7$; $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = x^3$, and $h\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $h(x) = \frac{1}{x^2}$.
   (a) What is $f \circ g$? What is $g \circ f$?
   (b) What is $f \circ h$? What is $h \circ h$?
   (c) In each case, what are the restrictions on the domain that are necessary? For example, is $f \circ h$ defined on all of $\mathbb{R}$? or some smaller subset?

5. We make the following definitions:

- let $\mathcal{E} = \{n \in \mathbb{Z} | n \equiv 0 \pmod 2\}$ be the set of even numbers,
- let $\mathcal{P} = \{n \in \mathbb{Z} | n \text{ is prime}\}$ be the set of primes,
- let $\mathcal{M}_3 = \{n \in \mathbb{Z} | n \equiv 0 \pmod 3\}$ be the set of multiples of three,
- and let $\mathcal{S}q = \{n \in \mathbb{Z} | \text{there exists } m \text{ such that } m^2 = n\}$ be the set of perfect squares.

(a) What is $\mathcal{E} \cap \mathcal{P}$?

(b) What is $\mathcal{E} \cap \mathcal{M}_3$?

(c) What is $\mathcal{E} \cap \mathcal{S}q$?

(d) What is $\mathcal{P} \cap \mathcal{S}q$?

(e) (difficult) For any k, define $\mathcal{M}_k = \{n \in \mathbb{Z} | n \equiv 0 \pmod k\}$ to be the set of multiples of k. In general, what is $\mathcal{M}_a \cap \mathcal{M}_b$?

6. Prove your answer to each part of the previous question.

7. Using the axioms and theorems from class, prove that $(a-b)(c-d) = (ac+bd)-(ad+bc)$.

8. Define the function $f \colon \mathbb{Z} \to \mathbb{Z}$ by $f(n) = n+1$, $g \colon \mathbb{Z} \to \mathbb{Z}$ by $g(n) = 2n$, and $h \colon \mathbb{Z} \to \mathbb{Z}$ by $h(n) = |n|$. (That is, $h(n) = n$ if $n \geq 0$, and $h(n) = -n$ if $n < 0$.)

(a) Is f one-to-one? Onto?

(b) Is g one-to-one? Onto?

(c) Is h one-to-one? Onto?

(d) Is $f \circ g$ equal to $g \circ f$? If so, prove it; if not, give a counterexample.

(e) Is $g \circ h$ equal to $h \circ g$? If so, prove it; if not, give a counterexample.