Math 145. Bezout's Theorem

Let k be an algebraically closed field. The purpose of this handout is to prove Bezout's Theorem and some related facts of general interest in projective geometry that arise along the way.

1. Setup and projective transformations

Let $f, g \in k[X, Y, Z]$ be homogenous of respective degrees $d, e \geq 1$, without any common irreducible factor. The zero loci $\underline{Z}(f)$, $\underline{Z}(g)$ have as their respective sets of irreducible components the zero loci of the (homogenous!) irreducible factors $\{f_i\}$ of f and $\{g_j\}$ of g due to:

Proposition 1.1. *If $f \in k[x_0, \dots, x_n]$ is homogeneous of degree $d > 0$ then all factors of f in $k[x_0, \dots, x_n]$ are homogeneous, and if f is irreducible then the closed set $\underline{Z}(f) \subset \mathbf{P}^n_k$ is irreducible of dimension $n - 1$.*

If g is another homogeneous irreducible polynomial in $k[x_0, \dots, x_n]$ then $\underline{Z}(f) = \underline{Z}(g)$ if and only if $f = cg$ for $c \in k^\times$.

Proof. If $f = f_1 f_2$ with non-constant f_j's then the degrees $d_j > 0$ of the f_j's must add up to d and the top-degree parts of the f_j's have product then must be the top degree part of f. But f is its own top-degree part (as f is homogeneous). By similar reasoning, the least-degree parts of the f_j's must have product equal to f and in particular degrees that add up to d. These least degrees are respectively $\leq d_1$ and $\leq d_2$, yet their sum is $d = d_1 + d_2$, so the least-degrees coincide with the top degrees for the f_j's; i.e., each f_j is homogeneous.

Next assume that f is irreducible. To show that $\underline{Z}(f)$ is irreducible, consider the surjective morphism

$$q : \mathbf{A}^{n+1} - \{0\} \to \mathbf{P}^n_k.$$

The preimage is the zero locus of f on $\mathbf{A}^{n+1}$ with the origin removed. If $\underline{Z}(f)$ is covered by two proper closed subsets in $\mathbf{P}^n_k$, then the zero locus of f in $\mathbf{A}^{n+1}_k - \{0\}$ would be covered by their proper closed preimages. That is, the zero locus of f in the open subset $\Omega = \mathbf{A}^{n+1}_k - \{0\} \subset \mathbf{A}^{n+1}_k$ would be *reducible*. But the zero locus of f in $\mathbf{A}^{n+1}_k$ is irreducible since f is irreducible, and it contains the origin as a proper closed subset (since the origin is of codimension $n+1 \geq 2$, so it cannot be an irreducible hypersurface), so its overlap with $\mathbf{A}^{n+1}_k - \{0\}$ is a non-empty open subset of an irreducible space. Such an open subset is always irreducible.

The zero locus V of f in $\mathbf{A}^{n+1}_k - \{0\}$ is irreducible of dimension n, and it maps onto $\underline{Z}(f)$ with all fibers of dimension 1. More specifically, each $q^{-1}(U_i) \to U_i$ is a surjective map between affine varieties with all fibers irreducible of dimension 1, so likewise whenever $\underline{Z}(f) \cap U_i$ is non-empty we see that

$$V \cap q^{-1}(U_i) \to \underline{Z}(f) \cap U_i$$

is a surjective map between affine varieties with all fibers of dimension 1. Thus, by the earlier handout on fiber dimension in the affine case, it follows that the target $\underline{Z}(f) \cap U_i$ has dimension $\dim(V \cap q^{-1}(U_i)) - 1 = n - 1$.

Finally, for irreducible homogenous $g \in k[x_0, \dots, x_n]$ such that $\underline{Z}(g) = \underline{Z}(f)$ we seek to prove that $g = cf$ for some $c \in k^\times$. Consider the preimages in $\mathbf{A}^{n+1}_k - \{0\}$. These preimages are the complements of the origin in the *irreducible* (!) zero loci of f and g in $\mathbf{A}^{n+1}_k$, so the closures of these preimages in $\mathbf{A}^{n+1}_k$ are those zero loci. That is, f and g have the same zero loci in $\mathbf{A}^{n+1}_k$, whence each is a multiple of the other in $k[x_0, \dots, x_n]$ by their irreducibility and the Nullstellensatz. This forces the multiplier relating them to be a unit in $k[x_0, \dots, x_n]$, which is to say an element of $c^\times$. ■

The hypotheses on f and g ensure that the *irreducible curves* $\underline{Z}(f_i)$, $\underline{Z}(g_j)$ are *distinct*, whence have intersection of smaller dimension, whence finite! For non-constant polynomials $h_1, h_2 \in k[u, v]$ and points P not on a common irreducible component of $\underline{Z}(h_1)$ and $\underline{Z}(h_2)$ (i.e., an isolated point of $\underline{Z}(h_1, h_2)$ or a point not in this common zero locus at all) our definition of the intersection number

$$I(P; h_1, h_2) = \dim_k(\mathscr{O}_{\mathbf{A}^2_k, P}/(h_1, h_2)) < \infty$$

(which vanishes if one of h_1 or h_2 is nonzero at P) is given in terms of the local rings (with $h_1, h_2 \in k[u, v] = \mathscr{O}(\mathbf{A}^2_k)$ having no common irreducible factor through P). Thus, we can define intersection numbers $I(P; f, g)$ for *homogenous* f, g and any point P by simply dehomogenizing f and g with respect to any of the homogenous coordinates not vanishing at P and then computing the (finite!) intersection number in the corresponding $\mathbf{A}_k^2$ via our affine theory. Note that if P has more than one non-vanishing coordinate then it does not matter which coordinate we use for dehomogenization: for example, if $P \in U_0 \cap U_2$ with $P = [1, a, b] = [1/b, a/b, 1]$ for $b \neq 0$ then we have an isomorphism

$$\mathscr{O}_{\mathbf{A}_k^2,(a,b)} = \mathscr{O}_{U_0,P} = \mathscr{O}_{\mathbf{P}_k^2,P} = \mathscr{O}_{U_2,P} = \mathscr{O}_{\mathbf{A}_k^2,(1/b,a/b)}$$

which carries the X-dehomogenizations of h_1 and h_2 over to the Z-dehomogenizations of h_1 and h_2 respectively, so passing to the respective quotients by these identifies (as local k-algebras with finite dimension as k-vector spaces) the two local ring quotients.

It is important to check that that this well-posed definition is unaffected by *arbitrary* projective linear change of coordinates, (not just the most trivial type which interchange the coordinates). That is, for $T \in \mathrm{PGL}_3(k)$ and $T' \in \mathrm{GL}_3(k)$ representing T, we claim that

$$I(P; f, g) = I(T^{-1}(P), f \circ T', g \circ T').$$

We give a computation-free proof below (as any other method appears tedious).

Remark 1.2. Intersection numbers can be defined entirely intrinsically — i.e., without making reference to coordinates or $\mathbf{A}^2$'s — within the framework of schemes. The fact that we built our theory out of reduced k-algebras forces us to use the above ad hoc approach, roughly because we cannot define the geometric object $\underline{Z}(f)$ in such a way that it "knows" the multiplicites of the irreducible factors of f (if such multiplicities occur which are > 1).

Lemma 1.3. *The above definition of $I(P; f, g)$ is unaffected by projective linear change of coordinates.*

Proof. From the affine case, it follows that our definition is "additive" with respect to taking products in f and g. That is, if $f = c \prod f_i^{r_i}$ and $g = c' \prod g_j^{s_j}$ is the decomposition into irreducible (homogenous!) factors (with $c, c' \in k^\times$), then

$$I(P; f, g) = \sum_{i,j} r_i s_j I(P; f_i, g_j)$$

(where the only non-zero terms on the right side are the ones for which $f_i(P) = g_j(P) = 0$). Since the formation of the irreducible factorization behaves well under linear homogenous change of coordinates, we are therefore reduced to studying the case in which f and g are *irreducible.* The point of this step is that the closed *subvarieties* $\underline{Z}(f)$ and $\underline{Z}(g)$ "remember" the irreducible f and g, by Proposition 1.1.

For irreducible homogenous f and g which aren't scalar multiples of each other (i.e., $\underline{Z}(f) \neq \underline{Z}(g)$), we now give a description of $I(P; f, g)$ which depends *only* on the closed subvarieties $\underline{Z}(f), \underline{Z}(g) \hookrightarrow \mathbf{P} = \mathbf{P}_k^2$ and not on any choice of coordinate system. In general, let Σ be any irreducible surface (i.e., 2-dimensional variety), and C, C' two distinct irreducible curves on Σ, so $C \cap C'$ is a finite set. Choose $P \in C \cap C'$, and consider the *surjections* $\mathscr{O}_{\Sigma,P} \twoheadrightarrow \mathscr{O}_{C,P}, \mathscr{O}_{C',P}$ (why are these surjective?). Let I_P, I'_P denote the respective kernels, and define $J_P = I_P + I'_P$, which is morally the "ideal defining $C \cap C'$" at P (except that this ideal might not be radical, which corresponds to the possibility of an intersection number being > 1).

For example, if $\Sigma = \mathbf{A}_k^2$ and $C = \underline{Z}(f)$, $C' = \underline{Z}(f')$ for irreducible f, f' and $P = (0, 0)$, then $I_P = f\mathscr{O}_{\mathbf{A}_k^2,0}$, $I'_P = f'\mathscr{O}_{\mathbf{A}_k^2,0}$, and $J_P = (f, f')\mathscr{O}_{\mathbf{A}_k^2,0}$. This example naturally leads us to consider the quotient k-algebra

$$\mathscr{O}_{\Sigma,P}/J_P.$$

In the case of distinct irreducible curves in $\mathbf{A}_k^2$ passing through a point, this quotient k-algebra is *exactly* the one whose k-dimension is the finite integer which we defined to be $I(P; f, g)$! We conclude that with $\Sigma = \mathbf{P}_k^2$, $C = \underline{Z}(f)$, $C' = \underline{Z}(g)$ where f and g are homogenous and irreducible (whence have *irreducible* dehomogenizations with respect to any of the coordinate variables which is non-vanishing at a point $P \in C \cap C'$), the *intrinsic* k-algebra $\mathscr{O}_{\Sigma,P}/J_P$ is of finite k-dimension equal to the coordinate-dependently defined $I(P; f, g)$. The intrinsic nature of this local ring quotient (its definition involves no specification of "coordinate", homogenous or otherwise!) thereby implies that this integer is in fact unaffected by arbitrary projective linear change of coordinates on $\mathbf{P}_k^2$.

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2. Main result

Here is Bezout's theorem:

Theorem 2.1. *For $f, g \in k[X, Y, Z]$ homogenous of respective positive degrees d, e and without any common irreducible factor,*

$$\sum_{P \in \mathbf{P}^2_k} I(P; f, g) = de.$$

We emphasize that there are only finitely many non-zero terms in the sum, corresponding to the *finite* set $\underline{Z}(f) \cap \underline{Z}(g)$.

Proof. By Lemma 1.3, we may make any linear homogenous change of coordinates. Also, using additivity of both intersection numbers and degrees with respect to products in f and g (as in the proof of Lemma 1.3), we immediately reduce to the case where f and g are *irreducible*. Since $\underline{Z}(f) \cap \underline{Z}(g)$ is a finite set, we may find a line $\mathbf{P}^1_k \hookrightarrow \mathbf{P}^2_k$ disjoint from this finite set (how?), and may then apply a projective linear change of coordinates so that this line is the line at infinity, $Z = 0$. That is, we can assume the intersection points all lie in the "ordinary affine plane" $\{Z \neq 0\} = U_2$.

Let $S = k[X, Y, Z]$, on which we have a natural direct sum decomposition

$$S = \oplus_{r \geq 0} S_r$$

with S_r the space of homogeneous polynomials of degree r. An ideal $I \subset S$ is called *homogeneous* if $I = \oplus I_r$ with $I_r = I \cap S_r$; i.e., I contains all homogeneous parts of all of its elements. The quotient $S/I = \oplus_{r \geq 0}(S_r/I_r)$ by a homogeneous ideal inherits a natural notion of "homogeneous part" (namely, the spaces S_r/I_r, or equivalently the image of S_r in S/I for each $r \geq 0$). It is clear that a sum of homogeneous ideals is homogeneous (check!), and that if $h \in S$ is homogeneous then the principal ideal $(h) = hS$ is homogeneous (why?). Thus, (f), (g), and $(f, g) = (f) + (g)$ are homogeneous ideals, so each of the quotients $S/f, S/g, S/(f, g)$ has a natural notion of "homogeneous parts".

The homogeneous parts of each are quotients of the S_r's and so have *finite* k-dimension as vector spaces. Since f and g are irreducibles in the UFD S and are not unit multiples of each other, the sequence of k-vector spaces

$$0 \to S/g \xrightarrow{f} S/g \to S/(f, g) \to 0$$

is exact. Note that although the second map respects the notion of "degree of homogeneous part" induced from S, the first map shifts such degrees up by the degree d of the homogenous f. If we let $(\cdot)_r$ denote the degree-rth homogeneous part of the ring then we have an exact sequence

$$0 \to (S/g)_{r-d} \xrightarrow{f} (S/g)_r \to (S/(f, g))_r \to 0$$

(where the pieces in negative degrees are understood to be 0). Thus,

$$\dim_k(S/(f, g))_r = \dim_k(S/g)_r - \dim_k(S/g)_{r-d}.$$

Using the exact sequence

$$0 \to S \xrightarrow{g} S \to S/g \to 0$$

and similar considerations with grading (where now the first map shifts up degrees by the degree e of g), we deduce

$$\dim_k(S/g)_n = \dim_k S_n - \dim_k S_{n-e}.$$

But for $m \geq 0$ we have $\dim_k S_m = m(m-1)/2$. Note that this applies to S_{n-e} only for $n \geq e$. Inserting this into the above formulae, we deduce

$$\dim_k(S/(f, g))_r = de$$

for r sufficiently large (more precisely, for $r \geq e + d$). The idea is now to show that for r sufficiently large, $(S/(f, g))_r$ has dimension equal to the sum of the intersection numbers. Roughly speaking, we want to view $S/(f, g)$ as the "coordinate ring" of the "necessarily affine closed subscheme"

$$\underline{Z}(f) \cap \underline{Z}(g) \subseteq \mathbf{P}^2_k - \{Z = 0\} \simeq \mathbf{A}^2_k,$$

but to make this precise we will need to be attentive to the fact that elements of S are not really functions on $\mathbf{P}_k^2$ and moreover the correct way to define this intersection $\underline{Z}(f) \cap \underline{Z}(g)$ as a geometric object requires allowing nilpotents in the structure sheaf (this corresponds to points P with $I(P; f, g) > 1$). Nonetheless, once such a "scheme-theoretic" intersection is defined (which we will do here without requiring the concept of a scheme), the k-dimension of its coordinate ring ought to be that sum of intersection numbers. Anyway, this is the geometric motivation (which comes from thinking about schemes and modern intersection theory).

Now consider the dehomogenizations $f_Z, g_Z \in k[x, y]$. We claim that the sum

$$\sum_{P \in \mathbf{P}_k^2} I(P; f, g) = \sum_{P \in \mathbf{A}_k^2} I(P; f_Z, g_Z)$$

is equal to $\dim_k k[x, y]/(f_Z, g_Z)$. Since f_Z and g_Z have no common irreducible factor in $k[x, y]$, the method of proof in §2 of the handout on intersection numbers shows that the natural map of k-algebras

$$k[x, y]/(f_Z, g_Z) \simeq \prod_{f_Z(P) = g_Z(P) = 0} k[x, y]_{m_P}/(f_Z, g_Z).$$

For a common zero P of f_Z and g_Z, the intersection number $I(P; f_Z, g_Z)$ is exactly the k-dimension of the Pth factor in this direct product, so the sum of these k-dimensions over all such P is the common k-dimension of both sides of this isomorphism of k-algebras, which in turn is visibly $\dim_k k[x, y]/(f_Z, g_Z)$.

We have proven that the two sides of Bezout's theorem are equal to the k-dimensions of $k[x, y]/(f_Z, g_Z)$ and $(S/(f, g))_r$ (for large r) respectively. Now in general for any ideal I in $k[x, y]$, define $I^{\mathrm{h}} \subseteq S$ to be the radical homogenous ideal corresponding to the Zariski closure of $\underline{Z}(I) \subseteq \mathbf{A}_k^2 = \mathbf{P}_k^2 - \{Z = 0\}$ in $\mathbf{P}_k^2$. This closure meets the open $\mathbf{A}_k^2 \subseteq \mathbf{P}_k^2$ in $\underline{Z}(I)$, so I is exactly the set of $f \in k[x, y]$ whose Z-homogenization lies in I^{h}. What is more important is that if $F \in S$ and $ZF \in I^{\mathrm{h}}$ then $F \in I^{\mathrm{h}}$ (as is easy to check from the definition of I^{h} in terms of Z-homogenization and Z-dehomogenization).

The ring $A = k[x, y]/I$ has finite k-dimension, and I claim that any finite k-algebra B admits a filtration by ideals I_j with

$$0 = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_N = B$$

with $\dim_k I_j/I_{j-1} = 1$ for all $0 < j \leq N$. Indeed, since B breaks up into a finite product of finite local k-algebras, we only need to consider local B. If $B = k$ there's nothing to say. Otherwise, since $B/\mathfrak{m}_B = k$ (as k is algebraically closed!!), we must have $\mathfrak{m}_B \neq 0$, so there is some integer $n \geq 1$ such that $\mathfrak{m}_B^n \neq 0$, $\mathfrak{m}_B^{n+1} = 0$. Choose any non-zero $b \in \mathfrak{m}_B^n$, so the non-zero principal ideal (b) is killed by $\mathfrak{m}_B$ and hence is of dimension 1 as a k-vector space, Applying induction on k-dimension to B/b, we get the desired chain of ideals. Now fix such a rising chain of ideals $\{I_j\}$ in $A = k[x, y]/I$, each of k-codimension 1 in the next, so $\dim_k A = \sum \dim_k I_j/I_{j-1}$. The rising chain of ideals $\{I_j\}$ in A gives rise to a rising chain of *homogenous* ideals I_j^{h} in $S/(f, g)$ (here I am implicitly identified ideals in a quotient ring with certain ideals in the original ring), whence $(S/(f, g))_r$ has a filtration by the rising chain of subspaces $(I_j^{\mathrm{h}}/I_{j-1}^{\mathrm{h}})_r$. We will now prove that if r is sufficiently large, then

$$\dim_k (I_j^{\mathrm{h}}/I_{j-1}^{\mathrm{h}})_r = 1$$

for all j. Taking the sum over j (with large r) will then complete the proof.

More generally, if $J \subseteq K$ is an inclusion of ideals in $k[x, y]$ with $\dim_k K/J = 1$, we claim that

$$\dim_k (K^{\mathrm{h}}/J^{\mathrm{h}})_r = 1$$

for large r (and hence the same conclusion applies for sufficiently large r for any *finite* rising chain of such ideals, so we'd be done). Let $n \in N = K^{\mathrm{h}}/J^{\mathrm{h}}$ be a homogenous element represented by a homogenous element $F \in K^{\mathrm{h}}$ of some degree r. Since the multiplication map by Z on $N = K^{\mathrm{h}}/J^{\mathrm{h}}$ is *injective* (why?), we see that $F/Z^r = F_Z \notin J$ if and only if $F \notin J^{\mathrm{h}}$, which is equivalent to $n \neq 0$. Since every element in K has the form F_Z for some homogenous $F \in K^{\mathrm{h}}$ and $K \neq J$, we conclude that $N \neq 0$. Now choose homogenous $n \in N$ *non-zero* with minimal degree d (note that by its definition, N has vanishing graded terms in negative degrees) and choose a homogenous degree d representative F for n as above. For any

$d' \geq d$ and any homogenous $n' \in N_{d'}$ represented by a homogenous F', since K/J has k-dimension 1 with basis given by F_Z, we conclude that $F'_Z \equiv cF_Z \bmod J$ for some $c \in k$, whence

$$Z^m(Z^d F' - cZ^{d'} F) \in J^{\mathrm{h}}$$

for a large integer m. But this implies

$$F' \equiv cZ^{d'-d} F \bmod J^{\mathrm{h}}$$

(as multiplication by Z is injective on $K^{\mathrm{h}}/J^{\mathrm{h}}$), so $n' = cZ^{d'-d}n$ in $N_{d'}$. Thus, for $d' \geq d$ the *non-zero* (!) element $Z^{d'-d}n \in N_{d'}$ is a k-basis, so indeed $N_r = (K^{\mathrm{h}}/J^{\mathrm{h}})_r$ has k-dimension equal to 1 for all large r (i.e., $r \geq d$). This is what we needed to prove.

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