Math 145. Fiber dimension

Let $f : Z \to Z'$ be a map between affine varieties over an algebraically closed field k, with respective dimensions d and d'. We assume that $f(Z)$ is dense in Z'. As we saw in class, this forces the k-algebra map $f^* : k[Z'] \to k[Z]$ to be injective, so there is a corresponding map $f^* : k(Z') \to k(Z)$ between function fields; comparing transcendence degrees over k shows $d \geq d'$, as we'd expect from geometric pictures. Some examples were given in class to illustrate the kind of behavior that can occur for the fibers $f^{-1}(z')$ as we vary $z' \in Z'$. The purpose of this handout is to prove that for "most" z', the Zariski closed set $f^{-1}(z')$ is non-empty with *all* irreducible components of dimension $d - d'$. The proof we give for this "geometric form" of transitivity of transcendence degree illustrates once again the power of Noether normalization for allowing us to use geometric reasoning to reduce to studying affine spaces, provided we have enough algebraic technique.

Theorem 0.1. *If f has dense image then $d \geq d'$ and for some non-empty open U' in Z' the fibers of f over $z' \in U'$ have dimension exactly $d - d'$. The non-empty fibers of f over $z' \in Z'$ have all irreducible components of dimension at least $d - d'$.*

Corollary 0.2. *For z' in some non-empty open of Z', the fibers have all of their irreducible components with dimension $d - d'$.*

This illustrates the basic guiding principle that what happens on the level of function fields (e.g., transitivity of transcendence degree) describes what happens geometrically over a suitable dense open.

Proof. (of theorem) Let us first explain why it suffices to prove *in general* the weaker assertion in which the final part of the theorem is relaxed to just require that all non-empty fibers merely have total dimension at least $d - d'$, allowing for the possibility that some irreducible components of these fibers might a priori have dimension less than $d - d'$. Grant this weaker-looking claim (again, we assume it to be proven in general, not just in one specific case!). We have to rule out the possibility that there is some $z' \in Z'$ with the closed set $f^{-1}(z')$ in Z non-empty with an irreducible component T of dimension strictly smaller than $d - d'$. Assuming there is such a z', let $\{T_i\}$ be the irreducible components of $f^{-1}(z')$, with $T_1 = T$. There is at least one other T_j (since the non-empty fiber $f^{-1}(z')$ has dimension at least $d - d'$), so

$$V = f^{-1}(z') - \bigcup_{j>1} T_j$$

is a non-empty open subset of $f^{-1}(z')$ lying entirely inside of the irreducible affine variety $T = T_1$. Thus, V has dimension equal to that of T, namely an integer less than $d - d'$.

But the closed set $f^{-1}(z')$ inherits the induced topology from Z (recall the definition of the Zariski topology), so there is an open W in Z with $W \cap f^{-1}(z') = V$. Choose a basic open Z_a in Z around a point of V with $Z_a \subseteq W$. We can identify Z_a with an affine variety of dimension d in such a way that the map

$$g : Z_a \subseteq Z \to Z'$$

is the geometric map corresponding to the injection $k[Z'] \to k[Z]_a$ (see HW 3, Exercise 5(i)). But now g is a map between affine varieties of dimensions d and d' yet there is a fiber $g^{-1}(z') = f^{-1}(z') \cap Z_a$ which is non-empty and open in the irreducible $(d - d')$-dimensional V, so $g^{-1}(z')$ has dimension strictly smaller than $d - d'$. This is a contradiction, as we assumed the weaker theorem had been proven *in general* (e.g., including the case of the newly constructed map g). We conclude that it is indeed enough to weaken the final condition of the theorem to just the total dimension of the non-empty fibers being at least $d - d'$.

Now we begin the proof. Let us now show in general that the image of f contains a non-empty open over which the fibers have dimension exactly $d - d' \geq 0$ (again, we are not proving directly that *all* irreducible components of these fibers have dimension $d - d'$). We argue by induction on $d - d'$. The case $d - d' = 0$ is HW 5, Exercise 2(ii). Suppose now that $d - d' > 0$ and the result is known for all smaller values. This will allow us to "throw away" closed sets. Pay careful attention to how we do this; it illustrates the importance in algebraic geometry of setting up proofs in extreme generality so as to make possible induction arguments on dimension (as one is often faced with auxiliary abstract closed sets of lower dimension which can be "discarded" if we had a very general inductive hypothesis).

For the multiplicative set $S = k[Z'] - \{0\}$, we see that $S^{-1}k[Z]$ is a finitely generated domain over the infinite field $S^{-1}k[Z'] = k(Z')$, so by Noetherian normalization, we can find a finite injection

$$k(Z')[T_1, \dots, T_r] \hookrightarrow S^{-1}k[Z]$$

over $k(Z')$ for some integer r, and upon comparing transcendence degrees of fraction fields (over k and over $k(Z')$) we see $r = d - d' \geq 1$. Composing with a $k(Z')$-algebra automorphism of $k(Z')[T_1, \dots, T_{d-d'}]$ which multiplies the T_j's by suitable elements of S, we may suppose that the image of each T_j lands in $k[Z]$ (just "clear denominators"), so we get an injective map

$$k[Z'][T_1, \dots, T_{d-d'}] \hookrightarrow k[Z]$$

of $k[Z']$-algebras which becomes *finite* after inverting some non-zero $a' \in k[Z']$ (by HW 5, Exercise 2(ii)).

Geometrically, our map $Z \to Z'$ factors as a composite

$$Z \xrightarrow{\pi} Y \xrightarrow{\ell} Z'$$

where $k[Y] = k[Z'][T_1, \dots, T_{d-d'}]$ over $k[Z']$ with ℓ the canonical projection, π has dense image containing an open set (by the "$d - d' = 0$" case), and for some non-zero $a' \in k[Z']$ the map π becomes finite (and thus surjective!) when we look at $Z_{a'} \to Y_{a'}$, using HW 3, Exercise 5 (i). By HW 5, Exercise 2 (ii) there is a non-zero $g \in k[Y]$ so that $Y_g \subseteq Y_{a'}$ and over Y_g the map π becomes finite *free* (i.e., $k[Y]_g \to k[Z]_g$ is finite free). If we write the non-zero $g \in k[Y]$ as a polynomial over $k[Z']$, consider any one of its non-zero coefficients. Replacing a' by its multiple by such a coefficient, we may assume that for all $z' \in Z'_{a'}$, the polynomial

$$g(z') \in k[T_1, \dots, T_{d'}]$$

obtained by "evaluating" the $k[Z']$-coefficients of g at z' has a non-zero coefficient and so is *non-zero* (note that "evaluation at z'" is the inverse of the natural isomorphism $k \simeq k[Z']/\mathfrak{m}_{z'}$).

Thus, for all $z' \in Z'_{a'}$ we have that $\ell^{-1}(z')_g$ is a non-empty open in the affine $(d - d')$-space $\ell^{-1}(z')$. Observe also that $\ell^{-1}(z') \subseteq Y_{a'}$, with π finite over $Y_{a'}$, so looking over the closed subset $\ell^{-1}(z')$ in $Y_{a'}$ we see that the map of affine algebraic sets

$$\pi : f^{-1}(z') = \pi^{-1}(\ell^{-1}(z')) \to \ell^{-1}(z')$$

is also finite for $z' \in Z'_{a'}$. But now look over the *non-empty* open $\ell^{-1}(z')_g$ in the affine space $\ell^{-1}(z')$. By the definition of g, the map $Z_g \to Y_g$ is *finite free*, so looking over the closed subset $\ell^{-1}(z')_g \subseteq Y_g$ we see that $\pi : f^{-1}(z')_g \to \ell^{-1}(z')_g$ arises from passing to reduced quotients on a map of affines which is finite free. The justification of this uses that "inverting an element" is compatible with both passage to quotient rings (HW 2, Exercise 3 (iii)) and to quotients by the nilradical (equivalently, A_a is reduced when A is reduced: if $x \in A$ and $x^n = 0$ in A_a with $n > 0$ then some $a^m x^n = 0$ in A, so $(ax)^{mn} = 0$ in the reduced A and hence $ax = 0$ in A, forcing $x = 0$ in A_a), as well as the fact that the identification of a basic open with an affine algebraic set as in HW 3, Exercise 5 (i).

But $\ell^{-1}(z')_g$ is a non-empty basic open in an affine $(d - d')$-space, hence is irreducible with dimension $d - d'$. Now we can deduce that the open $f^{-1}(z')_g$ in $f^{-1}(z')$ has *all* irreducible components with dimension $d - d'$, upon recalling from HW5, Exercise 1(iii) that if $V' \to V$ is a map of affine algebraic sets with $k[V]$ a domain of dimension δ and $k[V']$ a nonzero finite free $k[V]$-module then *all* irreducible components of V' have dimension δ.

We have shown that for all $z' \in Z'_{a'}$, the fiber $f^{-1}(z')$ contains an open set $f^{-1}(z')_g$ of dimension $d - d'$. Thus, the closure of this open set in $f^{-1}(z')$ has dimension $d - d'$. To prove that these fibers $f^{-1}(z')$ over $z' \in Z'_{a'}$ have dimension exactly $d - d'$, it suffices to consider the closed complement C in Z of Z_g and to show that $C \cap f^{-1}(z')$ has dimension less than $d - d'$ (at least if we replace a' by some non-zero multiple).

Since C is a *proper* closed subset of Z, the finitely many irreducible components $\{C_i\}$ of C are all affine varieties with dimension *less* than d. Thus, for each i, either C_i fails to have dense image in Z' and so $U'_i = Z' - \overline{f(C_i)}$ is a non-empty open, or else (by *induction*!) C_i has dense image in Z' and there is a non-empty open U'_i in Z' over which $C_i \to Z'$ has all fibers with dimension $\dim C'_i - d' < d - d'$. Replacing a' by a suitable non-zero multiple, we may assume at the start that $Z'_{a'}$ lies inside of all such U'_i (a vacuous condition if C is empty). For any $z' \in Z'_{a'}$, the fiber then $f^{-1}(z')$ meets each C_i in a closed set which is either empty or at worst of dimension less than $d-d'$. We conclude that $f^{-1}(z')$ is non-empty with dimension exactly $d-d'$ for all $z' \in Z'_{a'}$.

What remains to be proven is that *all* non-empty fibers have dimension at least $d-d'$. Assume otherwise, with $f^{-1}(z')$ having dimension less than $d-d'$. Composing with a finite surjective map $p : Z' \to k^{d'}$, note that $(p \circ f)^{-1}(p(z'))$ is a *disjoint* union of the closed sets $f^{-1}(z'')$ for the finitely many (closed) points $z'' \in p^{-1}(p(z'))$. We may replace Z by a suitable basic open as near the beginning of the proof so as to put ourselves in a situation where $(p \circ f)^{-1}(p(z'))$ is irreducible of dimension strictly smaller than $d-d'$. Renaming $k^{d'}$ as Z' and $p \circ f$ as f, we may assume after linear change of coordinates that $Z' = k^{d'}$ and $f^{-1}(0)$ is irreducible with dimension less than $d-d'$. Arguing as near the beginning of the proof with Noether normalization, but ignoring the auxiliary a' and g, we have a factorization of f as maps with dense image

$$Z \xrightarrow{\pi} k^d \xrightarrow{\ell} k^{d'}$$

where ℓ is projection onto the last d' coordinates and $f^{-1}(0) = \pi^{-1}(\ell^{-1}(0))$ is *non-empty* with dimension less than $d-d'$. But $\ell^{-1}(0)$ is an affine space of dimension $d-d'$ cut out by the "coordinate functions" $T_1, \dots, T_{d'}$ from the coordinate ring of $k^{d'}$. Thus, we are in the following geometrically strange situation: Z is irreducible of dimension d and there are d' elements $t_1, \dots, t_{d'} \in k[Z]$ such that their common zero locus $f^{-1}(0)$ is *non-empty* and of dimension less than $d-d'$. This violates one's geometric intuition that imposing r equations should cause the dimension of an affine algebraic set to drop by at most r (if the zero locus is non-empty in the first place!). Actually, this intuition is incorrect on reducible spaces with components of different dimensions.

By an induction on the number of equations, it suffices to prove the fundamental geometric fact about dimension in the lemma below. This fact is very interesting in it's own right and is a special case of Krull's Hauptidealsatz ("principal ideal theorem"), one of the most important theorems in commutative algebra. ■

Lemma 0.3. *Let Z be an affine algebraic set all of whose irreducible components have dimension d (e.g., Z irreducible of dimension d). Let $a \in k[Z]$ be a non-unit (so $d > 0$). Then the non-empty closed set $\underline{Z}(a) \subseteq Z$ has all of its irreducible components of dimension at least $d-1$.*

Proof. It is sufficient to treat the irreducible components of Z separately (and we can ignore the components on which a has no zeros). Thus, we may assume Z is irreducible. Suppose that the proper closed set $\underline{Z}(a)$ has an irreducible component of dimension strictly less than $d-1$. By replacing Z with a basic open that meets $\underline{Z}(a)$ in this component but is disjoint from the other irreducible components of $\underline{Z}(a)$, we may suppose that $\underline{Z}(a)$ is of dimension less than $d-1$.

Now by Noether normalization we can find a finite surjection $\pi : Z \to k^d$, so $\pi(\underline{Z}(a))$ is a closed set in k^d of dimension less than $d-1$. Now consider a monic polynomial $f = \sum c_j X^j \in k[T_1, \dots, T_d][X]$ for which $f(a) = 0$ in $k[Z]$. More specifically, since $k[T_1, \dots, T_d]$ is integrally closed, we may take f to be the *minimal* polynomial of a over the *field* $k(T_1, \dots, T_d)$. The "constant term" $c_0 \in k[T_1, \dots, T_d]$ is non-zero (as $a \neq 0$) and c_0 is certainly non-constant (i.e., not in $k^\times$) or else the condition $f(a) = 0$ would provide a multiplicative inverse to a in $k[Z]$, namely

$$(-1/c_0)(a^{n-1} + \pi^*(c_{n-1})a^{n-2} + \dots + \pi^*(c_1)),$$

contadicting the *non-emptiness* of $\underline{Z}(a)$. Thus, $\underline{Z}(c_0)$ has dimension $d-1$, so the preimage of this under the finite surjective map π has dimension at least $d-1$ (though it could well be reducible).

The idea at this point is that $\underline{Z}(c_0)$ of dimension $d-1$ should not have dimension less than that of $\underline{Z}(a)$, because, at least if f split completely over $k[Z]$ with roots $a = a_1, \dots, a_n$, then c_0 (or rather, $\pi^*(c_0)$) is the product of these roots (up to sign) and Galois theory should move each a_j to $a_1 = a$ and hence move each $\underline{Z}(a_j)$ to $\underline{Z}(a)$. This would tell us that *all* of the $\underline{Z}(a_j)$'s have dimension less that $d-1$, yet their union is $\underline{Z}(\pi^*(c_0)) = \pi^{-1}(\underline{Z}(c_0))$, which has dimension $d-1$. The problem with carrying out this idea is that $k[Z]$ might not contain such a set of roots and there's no reason to beleive that $k[Z]$ is stable under Galois theory automorphisms of $k(Z)$. In order to take care of this, we will need to replace $k(Z)$ by a suitable "Galois closure" and replace $k[Z]$ by an integral closure in there (while at the same time not destroying the running hypotheses).

Let $K/k(T_1, \dots, T_d)$ be a splitting field for the minimal polynomials of a finite set of generators for $k(Z)$ over $k(T_1, \dots, T_d)$, so this is a normal extension (i.e., purely inseparable over a Galois extension). Let R be the integral closure of $k[T_1, \dots, T_d]$ in K, so by the integral closure handout R is *finite* over $k[T_1, \dots, T_d]$ and of course contains $k[Z]$. Thus, R has dimension d (look at function fields) and $k[Z] \hookrightarrow R$ corresponds to a finite surjective map of affine varieties $Z' \to Z$. In particular, each irreducible component of $\underline{Z}(aR)$ maps onto an irreducible closed set in $\underline{Z}(a)$ via a *finite* map, so it follows (why?) that all such irreducible components have dimension less than $d-1$. Thus, we may replace Z by Z' and so can assume that $k(Z)$ is a normal extension of $k(T_1, \dots, T_d)$ and $k[Z]$ is the integral closure of $k[T_1, \dots, T_d]$ in $k(Z)$. But Galois theory tells us (since $k(Z)$ is a normal extension of $k(T_1, \dots, T_d)$, but perhaps not Galois if we are in positive characteristic) that the minimal polynomial f of a over $k(T_1, \dots, T_d)$ splits completely in $k(Z)$ with roots $a = a_1, \dots, a_n$ and there exist *automorphisms* of K over $k(T_1, \dots, T_d)$ moving each a_j to a_1. Such automorphisms must bring the integral closure $k[Z]$ back to itself (why?), so we get automorphisms of Z moving each $\underline{Z}(a_j)$ to $\underline{Z}(a_1)$ and thus *all* of these zero loci have dimension less than $d-1$. Thus, $\underline{Z}(\prod a_j)$ has dimension less than $d-1$. But $\prod a_j = (-1)^n c_0$, so $\pi^{-1}(\underline{Z}(c_0)) = \underline{Z}(\pi^*(c_0))$ has dimension less than $d-1$, contradicting the previous paragraph. ■