Math 145. Intersection numbers

1. Motivation

Let $f, g \in k[x,y]$ be nonconstant elements which vanish at a common point $\xi \in k^2$. In class we saw that ξ is an isolated point of $\underline{Z}(f,g)$ if $\mathscr{O}_{k^2,\xi}/(f,g)$ is finite-dimensional over k (in which case the dimension of this local quotient was defined to be the *intersection number* $I_\xi(f,g)$ of f and g at ξ). We emphasize that $\mathscr{O}_{k^2,\xi}/(f,g)$ depends on f and g, not just their radicals. Such generality in f and g when defining intersection numbers (i.e., allowing them to be reducible and even to have repeated irreducible factors) is essential for a sufficiently robust theory of intersection numbers even for *irreducible* plane curves, since the computation of $I_\xi(f,g)$ generally involves changing f modulo g yet $f+gh$ is often reducible (or has repeated irreducible factors) even when f and g are irreducible.

The aim of this handout is to show the converse result that if $\mathscr{O}_{k^2,\xi}/(f,g)$ is finite-dimensional over k then necessarily ξ is an isolated point of $\underline{Z}(f,g)$, and moreover that in such cases with $\xi = (0,0)$ the natural map of local rings

$$\mathscr{O}_{k^2,(0,0)}/(f,g) \to k[\![x,y]\!]/(f,g)$$

is an isomorphism. (On the left side (f,g) denotes the ideal generated by f and g in the "algebraic" local ring $\mathscr{O}_{k^2,(0,0)}$ whereas on the right side (f,g) denotes the ideal generated by f and g in the formal power series ring $k[\![x,y]\!]$. The context should make the intended meaning clear.) This isomorphism can be expressed for any $\xi = (a,b)$ via a linear change of variables:

$$\mathscr{O}_{k^2,\xi}/(f,g) \simeq k[\![x-a, y-b]\!]/(f,g)$$

using formal power series in $x-a$ and $y-b$. (Note that when f and g are expanded in terms of $x-a$ and $y-b$, the expansions have "constant terms" $f(a,b)$ and $g(a,b)$ that vanish.)

Our methods will be entirely algebraic because we have to use a lot of nilpotent elements and in classical algebraic geometry there is no geometric object attached to a non-reduced k-algebra (such as $k[t]/(t^5)$). Within the framework of schemes the arguments below can be expressed in a more geometric style. For example, although we only defined local rings at points on irreducible affine algebraic sets (since we only developed a theory of localization at primes for domains), within the framework of schemes it turns out that the ring $\mathscr{O}_{k^2,\xi}/(f,g)$ which we use a lot below is in fact the local ring at ξ on the "scheme" defined by $f = g = 0$ (*without* passing to radicals, as in the classical theory).

2. Results on quotients of algebraic local rings

Fix non-constant $f, g \in k[x,y]$ and choose a point $\xi \in \underline{Z}(f) \cap \underline{Z}(g) = \underline{Z}(f,g)$.

Proposition 2.1. *If ξ is an isolated point of $\underline{Z}(f,g)$ then $\mathscr{O}_{k^2,\xi}/(f,g)$ is finite-dimensional over k and it has nilpotent maximal ideal.*

Proof. The ideal (f,g) in $\mathscr{O}_{k^2,\xi}$ is unaffected by removing from f and g any irreducible factors that are non-vanishing at ξ, since such factors are units in the local ring at ξ. Thus, we can assume that *every* irreducible factor f_i of f and g_j of g vanishes at ξ. Hence, no f_i can be a $k^\times$-multiple of any g_j since otherwise $\underline{Z}(f,g)$ would contain an irreducible curve passing through ξ, contradicting the hypothesis that ξ is an isolated point of $\underline{Z}(f,g)$. In other words, $\underline{Z}(f,g)$ is *finite*. That is, the coordinate ring $k[x,y]/\mathrm{rad}(f,g)$ is finite-dimensional over k. The first key point is to upgrade this to the fact that $A := k[x,y]/(f,g)$ is also finite-dimensional over k.

Rather generally, if J is an ideal in a finitely generated algebra R over a field F and if $R/\mathrm{rad}(J)$ is finite-dimensional over F then R/J is also finite-dimensional over F. To see this, we first note that $\mathrm{rad}(J)$ has a finite set of generators, say $r_1, \dots, r_n$, since R is noetherian. There are integers $e_i > 0$ such that $r_i^{e_i} \in J$ for all i, so for $e = \max e_i$ we have $r_i^e \in J$ for all i. Hence, $(\sum x_i r_i)^{en} \in J$ for any $x_1, \dots, x_n \in R$ since every monomial in the multinomial expansion of this power will contain a factor of r_i^e for some i. This shows that $\mathrm{rad}(J)^N \subseteq J$ for $N = en$. (By the same argument, any ideal in any noetherian ring contains a power of its own radical.) Thus, in the quotient ring R/J the ideal $I = \mathrm{rad}(J)/J$ satisfies $I^N = 0$ with $R/I = R/\mathrm{rad}(J)$ finite-dimensional over F. To show that R is finite-dimensional over F, we just need to check finite-dimensionality over F for the successive quotients I^j/I^{j+1} of the descending chain of ideals $\{I^j\}_{j\geq 0}$ that terminates at (0) in finitely many steps. But each I^j is a finitely generated R-module (since I is finitely generated), so I^j/I^{j+1} is finitely generated as an R/I-module. Since R/I is finite-dimensional over F, it follows that I^j/I^{j+1} is finite-dimensional over F also. This completes the proof that R/J is finite-dimensional over F.

Returning to the situation of interest, $A = k[x,y]/(f,g)$ is finite-dimensional over k. By Exercise 1(iv) in HW6, there is a natural isomorphism of k-algebras $A \simeq \prod A_i$ where each A_i is local with nilpotent maximal ideal $\mathfrak{m}_i$. Killing nilpotents on both sides, we get

$$A/\mathrm{rad}(0) \simeq \prod A_i/\mathfrak{m}_i,$$

with each $k_i := A_i/\mathfrak{m}_i = k$ since A is finite-dimensional over k (so likewise for each A_i) and the projections $A \twoheadrightarrow k_i = k$ equal to evaluation at the points of $\underline{Z}(f,g)$. We claim that the maximal ideals of A must be exactly the $\mathfrak{m}_i$'s. Indeed, to compute the maximal ideals we can kill nilpotents, and the quotient $A/\mathrm{rad}(0) = \prod k_i = \prod_{i\in I} k$ has maximal ideals given by the projections to the evident factor rings k (indexed by I). Thus, ξ corresponds to one of the local factors A_{i_0}.

Letting $M_i \subset k[x,y]$ be the maximal ideal corresponding to $\mathfrak{m}_i$ (i.e., M_i is the kernel of $k[x,y] \twoheadrightarrow A \twoheadrightarrow A_i \to A_i/\mathfrak{m}_i = k$), it suffices to show that $k[x,y]_{M_i}/(f,g) \simeq A_i$ (since each A_i is finite-dimensional over k and has nilpotent maximal ideal). More precisely, we claim that the natural surjective map $q_i : k[x,y] \to k[x,y]/(f,g) = A \twoheadrightarrow A_i$ which kills f and g uniquely factors (as a ring map) through the localization map $\iota_i : k[x,y] \to k[x,y]_{M_i}$ and that the resulting map $k[x,y]_{M_i} \to A_i$ which kills f and g induces an isomorphism $k[x,y]_{M_i}/(f,g) \simeq A_i$.

Let $z_i \in k^2$ be the point corresponding to M_i. To show that q_i uniquely factors through ι_i, consider a hypothetical ring map $Q_i : k[x,y]_{M_i} \to A_i$ such that $Q_i \circ \iota_i = q_i$. For any fraction $h/H \in k[x,y]_{M_i}$ with $h, H \in k[x,y]$ and $H \notin M_i$ (i.e., $H(z_i) \neq 0$), applying Q_i to the identity $H \cdot (h/H) = h$ yields

$$q_i(H)Q_i(h/H) = Q_i(H)Q_i(h/H) = Q_i(H \cdot (h/H)) = Q_i(h) = q_i(h)$$

(suppressing the intervention of ι_i in the notation). By definition of $M_i = q_i^{-1}(\mathfrak{m}_i)$ we have $q_i(H) \notin \mathfrak{m}_i$ since $H \notin M_i$, so $q_i(H) \in A_i^{\times}$. In other words, the only *possible* value for $Q_i(h/H) \in A_i$ is $q_i(h)/q_i(H)$. It is straightforward to check that $h/H \mapsto q_i(h)/q_i(H)$ is well-defined (since if $h_1/H_1 = h_2/H_2$ with $H_1(z_i), H_2(z_i) \neq 0$ then $q_i(h_1)/q_i(H_1) = q_i(h_2)/q_i(H_2)$ in A_i since we can apply q_i to the identity $h_1H_2 = h_2H_1$ in $k[x,y]$), and likewise that this is a ring map. Hence, we have shown the existence and uniqueness of Q_i.

Finally, we have to show that the resulting maps $k[x,y]_{M_i}/(f,g) \to A_i$ are isomorphisms. These maps are surjective, since even $k[x,y] \to A_i$ is surjective. For injectivity, consider a fraction $h/H \in k[x,y]_{M_i}$ with $h, H \in k[x,y]$ and $H(z_i) \neq 0$ such that h/H maps to 0 in A. We want to show that h/H has vanishing image in $k[x,y]_{M_i}/(f,g)$. It is harmless to multiply by a unit, so we can scale through by $H \in k[x,y]_{M_i}^{\times}$ to pass to the case $H = 1$. That is, for $h \in k[x,y]$ we assume h has vanishing image in A_i and we want to show that h has vanishing image in $k[x,y]_{M_i}/(f,g)$. Note that it is harmless to change h modulo $(f,g) \subset k[x,y]$. Using the direct product decomposition

$$k[x,y]/(f,g) = A = \prod A_j,$$

the hypothesis on h is that its component in A_i vanishes. If $e_i \in A$ is the idempotent corresponding to A_i (i.e., its component in A_i is 1 and its component in the other factors A_j is zero) then multiplication by e_i $h \bmod (f,g)$ to itself in A. But e_i evaluations to 1 at z_i (i.e., its image in $A_i/\mathfrak{m}_i = k$ is 1), so a representative $\widetilde{e}_i \in k[x,y]$ of e_i lies *outside* M_i yet $\widetilde{e}_i h \in (f,g)$. Pushing this into the local quotient ring $k[x,y]_{M_i}/(f,g)$, the image of $\widetilde{e}_i$ is a unit which kills the image of h. Hence, the image of h in $k[x,y]_{M_i}/(f,g)$ vanishes, as desired. ■

Corollary 2.2. *If $\xi = (0,0)$ is an isolated point of $\underline{Z}(f,g)$ then the natural map*

$$\mathscr{O}_{k^2,(0,0)}/(f,g) \to k[\![x,y]\!]/(f,g)$$

is an isomorphism.

Proof. The crux of the matter is that (by the Proposition) the maximal ideal of $\mathscr{O}_{k^2,(0,0)}/(f,g)$ is *nilpotent*. Since this maximal ideal is generated by x and y (as $A_{\mathfrak{m}}$ has maximal ideal generated by $\mathfrak{m}$ for any noetherian domain A, such as $A = k[x,y]$), it follows that in $\mathscr{O}_{k^2,(0,0)}$ we have

$$(x,y)^N \subseteq (f,g)$$

for some large integer N. (These are ideals of $\mathscr{O}_{k^2,(0,0)}$, *not* ideals of $k[x,y]$!) This containment of ideals in the local ring $\mathscr{O}_{k^2,(0,0)}$ implies the same as ideals of the extension ring $k[\![x,y]\!]$, so we have equalities

$$\mathscr{O}_{k^2,(0,0)}/(f,g) = \mathscr{O}_{k^2,(0,0)}/(f,g,(x,y)^N), \quad k[\![x,y]\!]/(f,g) = k[\![x,y]\!]/(f,g,(x,y)^N).$$

Hence, to establish the desired isomorphism it suffices to show that the natural map

$$\varphi : \mathscr{O}_{k^2,(0,0)}/(x,y)^N \to k[\![x,y]\!]/(x,y)^N$$

is an isomorphism for any integer $N > 0$ (where the notation $(x,y)^N$ on each side denotes the Nth power of the ideal generated by x and y in the respective rings $\mathscr{O}_{k^2,(0,0)}$ and $k[\![x,y]\!]$).

Consider the maps of k-algebras

$$k[x,y]/(x,y)^N \to \mathscr{O}_{k^2,(0,0)}/(x,y)^N \xrightarrow{\varphi} k[\![x,y]\!]/(x,y)^N$$

whose composite map is the natural map

$$k[x,y]/(x,y)^N \to k[\![x,y]\!]/(x,y)^N$$

that is clearly an isomorphism (since formal power series can be uniquely expressed as an infinite sum of homogeneous parts, and the quotient by $(x,y)^N$ just kills the contribution in degrees $\geq N$). This shows that φ is surjective, and it reduces the injectivity of φ to the surjectivity of the map

$$k[x,y]/(x,y)^N \to \mathscr{O}_{k^2,(0,0)}/(x,y)^N.$$

Any fraction $h/H \in \mathscr{O}_{k^2,(0,0)}$ with $H(0,0) \neq 0$ can be expressed as $h \cdot (1/H)$, with h obviously hit by $k[x,y]/(x,y)^N$. Thus, we just need to hit $1/H$. For this purpose it suffices to show that H has unit image in $k[x,y]/(x,y)^N$ (as then the reciprocal of this image will be carried to the class of $1/H$ in $\mathscr{O}_{k^2,(0,0)}/(x,y)^N$, due to the uniqueness of multiplicative inverses in commutative rings). Since $c := H(0,0) \in k^\times$ and $H - c \in (x,y)$, $H = c + (H - c)$ is a unit by HW6, Exercise 1(i) (applied to the ring $k[x,y]/(x,y)^N$ and the ideal $I = (x,y)/(x,y)^N$ whose elements are nilpotent). ■