Math 145. MaxSpec

In this handout, we aim to prove two results. First, we prove a key fact which was used in the proof in class of the existence of the sheaf $\mathscr{O}_A$ on $\mathrm{Max}(A)$ for any reduced finitely generated k-algebra A. Then, with this sheaf in place, we show in a precise sense that A is functorially determined by the ringed space $\mathrm{MaxSpec}(A) := (\mathrm{Max}(A), \mathscr{O}_A)$ over k.

1. Gluing localizations

Let A be a reduced finitely generated k-algebra, and let $X = \mathrm{Max}(A)$. For each $a \in A$ the open subset $X_a = X - V_A((a))$ is naturally identified with $\mathrm{Max}(A_a)$ and A_a is thereby identified as a k-subalgebra of the k-algebra of functions on X_a. (Explicitly, since a is non-vanishing at every $x \in X_a$, the map $A_a \to \{X_a \to k\}$ carries b/a^n to the function $x \mapsto b(x)/a(x)^n$. See HW6 Exercise 4(i).)

By HW6 Exercise 4(ii) if $X_{a'} \subset X_a$ then a has unit image in $A_{a'}$ and the resulting unique A-algebra map $A_a \to A_{a'}$ is compatible with the restriction map carrying k-valued functions on X_a to k-valued functions on $X_{a'}$. In particular, for any $a, a' \in A$ we have $X_{aa'} = X_a \cap X_{a'}$, so fractions $f = b/a^n \in A_a$ and $f' = b'/{a'}^{n'} \in A_{a'}$ viewed as k-valued functions on X_a and $X_{a'}$ respectively agree on $X_a \cap X_{a'}$ if and only if f and f' have the same images in the common localization $A_{aa'} = (A_a)_{a'} = (A_{a'})_a$.

Theorem 1.1. *Let $f : X = \mathrm{Max}(A) \to k$ be a function, and choose $a_1, \dots, a_n \in A$ such that the open sets X_{a_i} cover X. If $f|_{X_{a_i}} \in A_{a_i}$ as a k-valued function on X_{a_i} for each i then $f \in A$ as a k-valued function on X.*

Proof. The condition that $\{X_{a_i}\}$ covers X is that their complements $V_A(a_i)$ have empty total intersection. But this intersection is $V_A((a_1, \dots, a_n))$, so the emptiness says that the ideal $(a_1, \dots, a_n)$ is not contained in *any* maximal ideals of A, which is to say that $(a_1, \dots, a_n) = (1)$.

In view of the discussion preceding the Theorem, we can now express the problem in terms of commutative algebra with A and its localizations: if $a_1, \dots, a_n \in A$ generate 1 and $f_i \in A_{a_i}$ are elements such that $f_i = f_j$ in $A_{a_i a_j}$ for all $i \neq j$ then there exists $f \in A$ such that $f = f_i$ in A_{a_i} for all i. To prove this, we write $f_i = b_i/a_i^{n_i}$ for all i. We may and do increase the exponent in each of these denominators (by multiplying the numerators accordingly) so that $n_i = n$ for all i. The equality $f_i = f_j$ in $A_{a_i a_j}$ says that the element $a_j^n b_i - a_i^n b_j$ of A vanishes in $A_{a_i a_j}$ for all i, j. That is, this difference is killed in A by $(a_i a_j)^{n_{ij}}$ for some $n_{ij} \geq 0$, and we may certainly increase these to all equal some common $m \geq 0$. Hence, for all $i \neq j$ we have

$$(a_i a_j)^m (a_j^n b_i - a_i^n b_j) = 0,$$

or equivaently

$$a_j^{n+m}(a_i^n b_i) - a_i^{n+m}(a_j^n b_j) = 0$$

in A for all $i \neq j$.

We arranged that $(a_1, \dots, a_n) = (1)$, so by raising an identity $\sum z_i a_i = 1$ to a large power we see that $(a_1^r, \dots, a_n^r) = (1)$ for all $r \geq 1$. Setting $r = n + m$, we get $c_i \in A$ such that

$$\sum c_i a_i^{n+m} = 1. \tag{1.1}$$

This should be viewed as a "partition of unity". For all $i \neq j$ we have

$$a_j^{n+m}(a_i^n b_i) = a_i^{n+m}(a_j^n b_j)$$

in A, so if we multiply both sides of (1.1) by $a_j^n b_j$ we get

$$a_j^n b_j = \sum_i c_i a_i^{n+m} a_j^n b_j = \sum_i c_i a_j^{n+m}(a_i^n b_i) = (\sum_i c_i a_i^n b_i) a_j^{n+m}.$$

Since $f_i = b_i/a_i^n = (b_i a_i^m)/a_i^{n+m}$, if we define $e = n+m$ and $b_i' = b_i a_i^m$ then $f_i = b_i'/a_i^e$ for all i and

$$b_j' = (\sum_i c_i b_i') a_j^e$$

in A for all j. Thus,

$$f_i = b_j'/a_j^e = \sum_i c_i b_i'$$

in A_{a_i} for all i. That is, the element $f = \sum_i c_i b_i' \in A$ satisfies $f = f_i$ in A_{a_i} for all i. ■

2. Geometric objects attached to rings

For any reduced finitely generated k-algebra A, we define the ringed space

$$\mathrm{MaxSpec}(A) = (\mathrm{Max}(A), \mathscr{O}_A)$$

over k attached to A. We know that the topological space $\mathrm{Max}(A)$ is contravariant in A, and we wish to upgrade this to a contravariance for $\mathrm{MaxSpec}(A)$. More importantly, once this is done, we will show that for a second reduced finitely generated k-algebra B, the resulting map of sets

$$\mathrm{Hom}_{k\text{-alg}}(A, B) \to \mathrm{Hom}(\mathrm{MaxSpec}(B), \mathrm{MaxSpec}(A))$$

is bijective: $\mathrm{MaxSpec}(A)$ *functorially* encodes A.

Lemma 2.1. *For any map of k-algebras $\phi : A \to B$, pullback of k-valued functions along the continuous map $\widetilde{\phi} : \mathrm{Max}(B) \to \mathrm{Max}(A)$ carries $\mathscr{O}_A(U)$ into $\mathscr{O}_B(\phi^{-1}(U))$ for all open $U \subset \mathrm{Max}(A)$. That is, $\widetilde{\phi}$ underlies a uniquely determined map of ringed spaces*

$$\mathrm{MaxSpec}(\phi) : \mathrm{MaxSpec}(B) \to \mathrm{MaxSpec}(A)$$

over k.

Proof. Choose $f \in \mathscr{O}_A(U)$. To show that $f \circ \widetilde{\phi} : \phi^{-1}(U) \to k$ lies in $\mathscr{O}_B(\widetilde{\phi}^{-1}(U))$, by the local nature of $\mathscr{O}_B$ we just have to find an open cover of $\widetilde{\phi}^{-1}(U)$ by subsets of $Y = \mathrm{Max}(B)$ of the form Y_{b_i} such that $(f \circ \widetilde{\phi})|_{Y_{b_i}} \in B_{b_i}$ for all i.

Since U is open in $X = \mathrm{Max}(A)$, it has an open cover by sets of the form X_{a_i}. The preimage of $V_A(a)$ under $\widetilde{\phi}$ is $V_B(\phi(a))$ since $a(\widetilde{\phi}(y)) = (\phi(a))(y)$, so $\widetilde{\phi}^{-1}(X_a) = Y_{\phi(a)}$ by passing to complements. Hence, the open sets $Y_{\phi(a_i)}$ cover $\widetilde{\phi}^{-1}(U)$, so it suffices to show that $(f \circ \widetilde{\phi})|_{Y_{\phi(a_i)}} \in B_{\phi(a_i)}$ for all i. But

$$(f \circ \widetilde{\phi})|_{Y_{\phi(a_i)}} = (f|_{X_{a_i}}) \circ \widetilde{\phi},$$

and $f|_{X_{a_i}} = f_i \in A_{a_i}$ for all i since $f \in \mathscr{O}_A(U)$. Thus, we are reduced to showing that pullback of k-valued functions along $\widetilde{\phi} : Y_{\phi(a)} \to X_a$ induces $\phi : A_a \to B_{\phi(a)}$. This is the problem of comparing two maps from A_a into the k-algebra of k-valued functions on $Y_{\phi(a)}$, and as such it suffices to compare after composing with $A \to A_a$ (due to the uniqueness aspect of the universal property of this localization).

Since $A \to A_a$ is induced by the restriction map along $X_a \hookrightarrow X$ on k-valued functions (HW6, Exercise 4(i)), our problem is to show that pullback of k-valued functions along the composite map $Y_{\phi(a)} \to X_a \to X$ carries A into $B_{\phi(a)}$ via ϕ. But this composite map of topological spaces is equal to the composite map

$$Y_{\phi(a)} \to Y \to X$$

(the latter being $\widetilde{\phi}$), and pullback of k-valued functions along $Y_{\phi(a)} \to Y$ induces the natural map $B \to B_{\phi(a)}$. Hence, we are finally reduced to checking that pullback of k-valued functions along $\widetilde{\phi} : Y \to X$ induces $\phi : A \to B$. In view of the *definition* of $\widetilde{\phi}$, this was a calculation early in our study of "polynomial maps" (in terms of a choice of presentation of each of A and B as quotients of polynomial rings over k). ■

It is clear that if $\psi : B \to C$ is a second such map then

$$\mathrm{MaxSpec}(\psi \circ \phi) = \mathrm{MaxSpec}(\phi) \circ \mathrm{MaxSpec}(\psi).$$

Hence, $A \rightsquigarrow \mathrm{MaxSpec}(A)$ is a contravariant functor from the category of reduced finitely generated k-algebras to the category of ringed spaces over k. The key fact is that this construction involves *no new maps* in the following sense:

Theorem 2.2. *The natural map of sets*

$$\mathrm{Hom}_{k\text{-alg}}(A, B) \to \mathrm{Hom}(\mathrm{MaxSpec}(B), \mathrm{MaxSpec}(A))$$

defined by $\phi \mapsto \widetilde{\phi}$ is bijective.

Proof. The map induced by pullback along $\widetilde{\phi}$ on global k-valued functions carries A into B via ϕ. Hence, $\widetilde{\phi}$ as a map between ringed spaces over k recovers ϕ! This proves the injectivity.

Now let $f : \mathrm{MaxSpec}(B) \to \mathrm{MaxSpec}(A)$ be an arbitrary map as ringed spaces over k. Pullback along f on global k-valued functions carries $A = \mathscr{O}_A(\mathrm{Max}(A))$ into $B = \mathscr{O}_B(\mathrm{Max}(B))$ via some k-algebra map ϕ. We therefore aim to prove that $f = \widetilde{\phi}$.

Since a map between ringed spaces over k is determined by the map on underlying topological spaces, it suffices to show that for $y \in Y = \mathrm{Max}(B)$, the point $x = f(y) \in X = \mathrm{Max}(A)$ is equal to $\widetilde{\phi}(y)$. Since a point of X is just a maximal ideal of A, and so is determined by the set of elements $a \in A$ that vanish at this point (this set being exactly the associated maximal ideal of A), we just have to show that $a(x) = 0$ if and only if $a(\widetilde{\phi}(y)) = 0$. But $a(x) = a(f(y)) = (a \circ f)(y) = (\phi(a))(y)$ by definition of ϕ. By the computations from our early study of polynomial maps, we know that $(\phi(a))(y) = a(\widetilde{\phi}(y))$. Hence, $a(x) = a(\widetilde{\phi}(y))$ for all $a \in A$, so $x = \widetilde{\phi}(y)$ as desired. ■