Math 145. Noetherian conditions

Let R be a commutative ring. In class we saw that the ascending chain condition on ideals of R is equivalent to the condition that all ideals in R are finitely generated. The aim of this handout is to prove the equivalence of these two conditions with yet another "finiteness" condition, on the module theory of R:

Theorem 0.1. *If all ideals of R are finitely generated then every submodule of a finitely generated R-module is also finitely generated.*

In class we noted that the converse of this theorem is true (and elementary), since ideals of R are precisely submodules of the module $M = R \cdot 1$. The subtlety with the implication in the theorem is that if we try to induct on the number of generators of M then we run into the problem that this does *not* control the number of generators needed for submodules of M, in contrast with the case of PID's. (For example, the ideal (x, y) in the ring $R = k[x, y]$ does not have a single generator as an R-module, even though $M = R$ does have a single generator.) For this reason, we will not argue by induction on the number of generators. We will argue by induction in another way.

Proof. Let M be a finitely generated R-module, say with n generators. That is, there is a surjective map $q : R^n \twoheadrightarrow M$. Hence, any submodule $M' \subseteq M$ is the image of a submodule $q^{-1}(M') \subseteq R^n$, so to show that M' is finitely generated it suffices to prove that $q^{-1}(M')$ is finitely generated. In other words, we are reduced to proving finite generation for submodules of each R^n. *This* we will prove by induction on n. The case $n = 1$ is precisely our hypothesis that all ideals of R are finitely generated!

Now assume $n > 1$ and that the case $n - 1$ is known. View R^{n-1} as a submodule of R^n via

$$(r_1, \ldots, r_{n-1}) \mapsto (r_1, \ldots, r_{n-1}, 0),$$

so $R^n/(R^{n-1}) = R$ via projection to the nth coordinate. For a submodule $N \subseteq R^n$, we get a submodule $N' = N \cap R^{n-1} \subseteq R^{n-1}$ and a quotient

$$N'' = N/N' \subseteq R^n/R^{n-1} \simeq R.$$

Thus, by induction N' is finitely generated, and by the base case N'' is finitely generated (we have even identified it with an ideal of R), so we have reached the following situation: we have a module N over R and a finitely generated submodule N' such that the quotient N/N' is also finitely generated. Then we claim that N is finitely generated.

Explicitly, suppose $\{e'_1, \ldots, e'_m\}$ is a generating set of N', and let $\{e_1, \ldots, e_r\}$ be a subset of N that lifts a generating set of N/N'. Then $\{e'_1, \ldots, e'_m, e_1, \ldots, e_r\}$ is a generating set of N (check!). ■

If you think about it, this method of proof essentially constructs a finite chain of submodules of M' so that the successive quotients in the chain are each identified with either an ideal of R or the quotient of R by an ideal. In this way, we see that the "ideal theory" of R exerts a tight control on the module theory over R. If you look back at how one proves the existence of bases of finite-dimensional vector spaces over a field or the structure theorem for modules over a PID, you'll see that those earlier arguments are essentially special cases of the kind of analysis we have just carried out (except that in those cases we get more precise structure theorems for the modules, so we have to do some more work which is not relevant over general noetherian R).