Math 145. Morphisms from quasi-projective varieties

1. Motivation

For homogeneous polynomials $F_0, \dots, F_m \in k[x_0, \dots, x_n]$ of the same degree $d > 0$ we wish to prove by a systematic method that the well-defined map of sets

$$\varphi : \mathbf{P}^n - \underline{Z}(F_0, \dots, F_m) \to \mathbf{P}^m$$

given by

$$[x_0, \dots, x_n] \mapsto [F_0(x), \dots, F_m(x)]$$

is a morphism. The principle is to take an algebro-geometric view of the map of sets

$$q : \mathbf{A}^{n+1} - \{0\} \to \mathbf{P}^n$$

given by $(x_0, \dots, x_n) \mapsto [x_0, \dots, x_n]$ in the definition of projective space. Note that $k^\times$-scaling on $\mathbf{A}^{n+1} - \{0\}$ preserves the fibers of q, so for any subset $S \subset \mathbf{P}^n$ the preimage $q^{-1}(S) \subset \mathbf{A}^{n+1} - \{0\}$ is stable under $k^\times$-scaling.

We claim that q really is a morphism of varieties, and that if $U \subset \mathbf{P}^n$ is any non-empty open set (so $q^{-1}(U)$ is open in $\mathbf{A}^{n+1} - \{0\}$) then for any morphism $f : q^{-1}(U) \to Y$ to an abstract algebraic set which is invariant under $k^\times$-scaling on $q^{-1}(U)$ the resulting well-defined map of sets $\overline{f} : U \to Y$ given by $[x_0, \dots, x_n] \mapsto f(x)$ is a morphism. By taking $U = \mathbf{P}^n - \underline{Z}(F_0, \dots, F_m)$, this would reduce the problem of whether or not φ above is a morphism to the analogous problem for the set map $\mathbf{A}^{n+1} - \{0\} \to \mathbf{P}^m$ (which we will analyze by working Zariski-locally on the source). In class we saw an application to the fact that the action on $\mathbf{P}^n$ by $\mathrm{PGL}_{n+1}(k)$ is through (auto)morphisms as a variety (not just as a set), which may seem messy to verify by bare hands since this action generally does not carry any of the "standard affine opens" $U_i = \{x_i \neq 0\}$ into any other.

2. The universal property of q

Let us first show that q is a morphism, and that it has the asserted mapping property for morphisms $f : q^{-1}(U) \to Y$ (with $U \subset \mathbf{P}^n$ a non-empty open set). The morphism property for q is local on the source in the sense that it suffices to check it on the constituents of an open cover of the source (since continuity and sheaf compatibility are local on the source) We work with the open affine cover $V_i = \{x_i \neq 0\}$ of $\mathbf{A}^{n+1} - \{0\}$ (for $0 \le i \le n$). The ith such open subset is carried into the open affine $\mathbf{A}^n = U_i \subset \mathbf{P}^n$, with

$$k[V_i] = k[x_0, \dots, x_n]_{x_i}, \quad k[U_i] = k[x_0/x_i, \dots, x_n/x_i].$$

There is an evident injective map

$$k[U_i] \hookrightarrow k[V_i],$$

and it is straightforward to check (do it!) that the induced morphism $V_i \to U_i$ set-theoretically recovers $q|_{V_i}$. Hence, q is indeed a morphism.

Of greater interest is the mapping property of q: if $U \subset \mathbf{P}^n$ is a non-empty open set and $f : q^{-1}(U) \to Y$ is a *morphism* to an abstract algebraic set Y such that f is invariant under the $k^\times$-scaling action on $q^{-1}(U)$ then we shall show that the induced map of sets $\overline{f} : U \to Y$ defined by $[x_0, \dots, x_n] \mapsto f(x)$ is a morphism. For each of the standard affine opens $U_i = \{x_i \neq 0\} \subset \mathbf{P}^n$, the overlaps $U \cap U_i$ are an *open cover* of U (with its open subspace structure from $\mathbf{P}^n$). For a set map between abstract algebraic sets to be a morphism it suffices to work locally on the source (i.e., on the constituents of an open covering), so it suffices to show that $\overline{f}|_{U \cap U_i}$ is a morphism for each i. That is, we may work over each $U \cap U_i$ separately, so we fix such an i_0 and work on $U \cap U_{i_0}$.

The map $q^{-1}(U_i) \to U_i = \mathbf{A}^n$ can be expressed in a rather concrete form (as we saw in the proof that U_i is a morphism), namely the map

$$\pi_i : U_i' = \{x \in \mathbf{A}^{n+1} \mid x_i \neq 0\} \to \mathbf{A}^n = U_i$$

between affine varieties arising from $q|_{U_i'}$ defined by

$$(x_0, \dots, x_n) \mapsto (x_0/x_i, \dots, x_n/x_i).$$

Note in particular that there is an "algebraic section": a morphism $s_i : \mathbf{A}^n \to U_i'$ given by $(t_0, \dots, \widetilde{t_i}, \dots, t_n) \mapsto (t_0, \dots, t_{i-1}, 1, t_{i+1}, \dots, t_n)$ such that $\pi_i \circ s_i = \mathrm{id}_{\mathbf{A}^n}$. (Algebraically, s_i corresponds to the k-algebra map $k[U_i'] = k[\mathbf{A}^{n+1}]_{x_i} \to k[t_0, \dots, \widetilde{t_i}, \dots, t_n]$ defined by $x_j \mapsto t_j$ for $j \neq i$ and $x_i \mapsto 1$.) In more geometric terms, we have an isomorphism of affine varieties

$$h_i : U_i' \simeq \mathbf{A}^n \times (\mathbf{A}^1 - \{0\})$$

given by $(x_0, \dots, x_n) \mapsto ((x_0/x_i, \dots, x_n/x_i), x_i)$ with inverse

$$((t_0, \dots, \widehat{t_i}, \dots, t_n), c) \mapsto (ct_0, \dots, ct_{i-1}, c, ct_{i+1}, \dots, ct_n),$$

and this carries $\pi_i : U_i' \to \mathbf{A}^n$ over to the standard projection $\mathbf{A}^n \times (\mathbf{A}^1 - \{0\}) \to \mathbf{A}^n$ from the direct product. In these terms, s_i corresponds to the "constant section" $\mathbf{A}^n \to \mathbf{A}^n \times (\mathbf{A}^1 - \{0\})$ defined by $t \mapsto (t, 1)$.

Returning to our problem of verifying that $\overline{f} : U \cap U_i \to Y$ is a morphism, we note that the isomorphism h_i carries $q^{-1}(U_i) = U_i'$ over to $U_i \times (\mathbf{A}^1 - \{0\})$ in such a way that π_i goes over to projection to the U_i-factor of the direct product and the $k^\times$-scaling action goes over to the usual $k^\times$-scaling action on the second factor $\mathbf{A}^1 - \{0\}$ of the direct product. Hence, this restricts to an isomorphism $q^{-1}(U \cap U_i) \simeq (U \cap U_i) \times (\mathbf{A}^1 - \{0\})$ carrying the $k^\times$-scaling action on $q^{-1}(U \cap U_i)$ over to the usual scaling action on the second factor of $(U \cap U_i) \times (\mathbf{A}^1 - \{0\})$ and carrying $q : q^{-1}(U \cap U_i) \to U \cap U_i$ over to the standard projection to the first factor of the direct product.

Our problem now takes on a more concrete form, as follows. For $V_i = U \cap U_i$, we are given a morphism $f_i : V_i \times (\mathbf{A}^1 - \{0\}) \to Y$ such that $f_i(v, c) = f_i(v, tc)$ for all $t \in k^\times$ and we aim to show that the well-defined map of sets $\overline{f}_i : V_i \to Y$ given by $v \mapsto f_i(v, t)$ for all $t \in k^\times$ is a morphism. But this is easy: just take $\overline{f}_i(v) = f_i(v, 1)$! That is, we recognize the set map $\overline{f}_i$ as the composition of the morphism f_i with the inclusion morphism $V_i \to V_i \times (\mathbf{A}^1 - \{0\})$ defined by $v \mapsto (v, 1)$ (corresponding to the k-algebra map $k[V_i][x, 1/x] \to k[V_i]$ via $x \mapsto 1$ when V_i is affine, and in general built from the analogue of this over open affines in V_i).

3. Application

To illustrate the mapping property of q, we now apply it to address the question raised at the outset, namely proving that φ is a morphism. The set map

$$f : \mathbf{A}^{n+1} - \underline{Z}(F_0, \dots, F_m) \to \mathbf{P}^m$$

given by

$$(x_0, \dots, x_n) \mapsto [F_0(x), \dots, F_m(x)]$$

is well-defined and visibly invariant under $k^\times$-scaling on the source (which is the q-preimage of the open complement of the locus of common zeros of the F_j's in $\mathbf{P}^n$), so it suffices to show that f is a morphism. It suffices to find an open cover $\{\Omega_\alpha\}$ of the source so that $f|_{\Omega_\alpha} : \Omega_\alpha \to \mathbf{P}^m$ is a morphism for each α.

We take our open cover to be the open sets $V_j = \mathbf{A}^{n+1} - \underline{Z}(F_j)$. This is carried by f into the *open* set $\{y_j \neq 0\}$ in $\mathbf{P}^m$ (where $y_0, \ldots, y_m$ denote the homogeneous "coordinates" on $\mathbf{P}^m$), so it suffices to show that each of the set maps

$$f_j : V_j \to \{y_j \neq 0\} = \mathbf{A}^m$$

given by

$$(x_0, \ldots, x_n) \mapsto (F_0(x)/F_j(x), \ldots, F_m(x)/F_j(x))$$

is a morphism. But V_j is affine with coordinate ring $k[x_0, \ldots, x_n]_{F_j}$ and the target $\{y_j \neq 0\} = \mathbf{A}^m$ of f_j is an affine space with coordinate ring $k[y_0/y_j, \ldots, y_m/y_j]$, so we just need to write down a map of k-algebras

$$k[y_0/y_j, \ldots, y_m/y_j] \mapsto k[x_0, \ldots, x_n]_{F_j}$$

that induces f_j. It is straightforward to check that the map

$$y_i/y_j \mapsto F_i(x)/F_j(x) \in k[x_0, \ldots, x_n]_{F_j}$$

does the job (check!).