Math 145. Smoothness for irreducible curves and hypersurfaces

1. Hypersurfaces

Let $V = \{f = 0\} \subset k^n$ for an irreducible $f \in k[t_1, \dots, t_n]$. We claim that the locus of smooth points $\xi \in V$ is a non-empty Zariski-open set; in particular, it is Zariski-dense. Put another way, we claim that the locus of non-smooth points in an irreducible affine hypersurface is a proper Zariski-closed subset of V. For example, in an irreducible plane curve this says that the set of non-smooth points is *finite*. (Using more powerful tools, it can be shown that these assertions for irreducible hypersurfaces are valid for any affine variety whatsoever. Our argument will use the "explicit" nature of hypersurfaces, so it does not generalize.)

As we have seen in class, a point $\xi \in V$ is smooth precisely when one of the partial derivatives $\partial_{t_i}(f)$ is non-zero at ξ. Hence, the non-smooth locus in $V = \underline{Z}(f)$ is the Zariski-closed set $\underline{Z}(f, \partial_{t_1} f, \dots, \partial_{t_n} f)$. This shows that the smooth locus in V is Zariski-open, but perhaps it is empty! It remains to show that V contains *some* smooth point (as then the proved Zariski-openness and the irreducibility of V takes care of the density aspects). We argue by contradiction: suppose that *every* point in V is non-smooth, which is to say that V is contained in the zero locus of the ideal generated by the $\partial_{t_i} f$'s. That is, we assume

$$\underline{Z}(f) \subseteq \underline{Z}(\partial_{t_1} f, \dots, \partial_{t_n} f).$$

Since $\underline{I}(V) = (f)$ by the Nullstellensatz (as f is irreducible in the UFD $k[t_1, \dots, t_n]$), it follows that

$$\mathrm{rad}(\partial_{t_1} f, \dots, \partial_{t_n} f) \subseteq (f),$$

so $\partial_{t_i} f \in (f)$ for all i. That is, $f | \partial_{t_i} f$ for all i. This sounds impossible, since $\partial_{t_i} f$ has lower t_i-degree than f (assuming that t_i appears in f), but keep in mind that in positive characteristic a nonzero polynomial can have vanishing derivative. More specifically:

Lemma 1.1. *Consider a non-constant $f \in k[t_1, \dots, t_n]$ such that t_{i_0} appears in f and $f | \partial_{t_{i_0}} f$. Then $\mathrm{char}(k) = p > 0$ and f involves t_{i_0} only through powers of $t_{i_0}^p$.*

Proof. By relabeling the variables, we can assume $i_0 = n$ and that f involves t_n (or else there is nothing to do). Consider the t_n-expansion

$$f = h_0 + h_1 t_n + \dots + h_d t_n^d$$

with $h_j \in k[t_1, \dots, t_{n-1}]$ and $h_d \neq 0$ with $d > 0$. Then

$$\partial_{t_n} f = h_1 + 2h_2 t_n + \dots + d h_d t_n^{d-1},$$

so in $k(t_1, \dots, t_{n-1})[t_n]$ we see that if $\partial_{t_n} f \neq 0$ then its t_n-degree is $< d$. Such a degree bound is incompatible with divisibility by f, so $\partial_{t_n} f = 0$. In other words, $j h_j = 0$ for all $1 \le j \le d$. In other words, if $h_j \neq 0$ then $j = 0$ in k. Since $h_d \neq 0$ we at least have $d = 0$ in k, so $\mathrm{char}(k) = p > 0$ (as $d > 0$). Hence, $p | j$ whenever $h_j \neq 0$. This says exactly that f lies in $k[t_1, \dots, t_{n-1}][t_n^p]$, as desired. ∎

Applying the Lemma to the irreducible f, since $f | \partial_{t_i} f$ for all i and *some* t_i does occur in f we see that necessarily $\mathrm{char}(k) = p > 0$ and every t_i appearing in f must appear only through powers of t_i^p. That is, under our assumption that $\underline{Z}(f)$ has no smooth points we see that necessarily $\mathrm{char}(k) = p > 0$ and $f \in k[t_1^p, \dots, t_n^p]$. Thus, for every monomial $\prod t_i^{e_i}$ appearing in f we have $p | e_i$ for all i (some e_i might be 0, but that is fine). Writing $e_i = p e_i'$, such a monomial is the pth power of $\prod t_i^{e_i'}$. The coefficient c of $\prod t_i^{e_i}$ appearing in f can also be written as c'^p for some $c' \in k$ since k is algebraically closed. We conclude that f is a sum of pth powers, so f itself is a pth power in $k[t_1, \dots, t_n]$ (as we are in characteristic p). This is a contradiction, since $p \ge 2$ and f is assumed to be irreducible in $k[t_1, \dots, t_n]$!

2. Curves

Let C be an irreducible affine curve. In class we asserted that C is smooth if and only if $k[C]$ is integrally closed, and the proof reduced to the following fact from commutative algebra that we now establish.

Proposition 2.1. *Let R be a 1-dimensional local noetherian domain with fraction field K. Then R is integrally closed in K if and only if R is a discrete valuation ring.*

The property of being integrally closed for higher-dimensional local noetherian domains has no concrete description akin to the notion of a discrete valuation ring in the 1-dimensional case.

Proof. Discrete valuation rings are UFD's, hence integrally closed, so the real content is the converse direction. Letting $\mathfrak{m}$ be the unique maximal ideal, since R is a 1-dimensional domain it follows that the prime ideals (0) and $\mathfrak{m}$ in R are distinct and must be the only prime ideals of R.

The first key point is that $\mathfrak{m}/\mathfrak{m}^2 \neq 0$. To prove this, assume to the contrary. Recall from Nakayama's Lemma that if M is a finitely generated module over a local ring $(A, \mathfrak{n})$ and $M/\mathfrak{n}M = 0$ then $M = 0$. This can be applied to $A = R$ and $M = \mathfrak{m}$ because the ideal $\mathfrak{m}$ of R is finitely generated as an R-module (since R is *noetherian*). Since $\mathfrak{m} \neq 0$, it follows that indeed $\mathfrak{m}/\mathfrak{m}^2 \neq 0$. We may therefore choose an element $t \in \mathfrak{m} - \mathfrak{m}^2$. Our goal is to show that $\mathfrak{m} = (t)$ (thereby establishing one of the equivalent conditions that defines a discrete valuation ring).

Suppose that the inclusion $tR \subseteq \mathfrak{m}$ is not an equality. We shall use this to construct an element of K not in R that is integral over R, so R is not integrally closed in K. Put another way, this will show that if R is integrally closed in K then necessarily $tR = \mathfrak{m}$, so R is indeed a discrete valuation ring.

In the quotient ring R/tR the nonzero prime ideal $P = \mathfrak{m}/tR$ is the only prime (since R has (0) and $\mathfrak{m}$ as its only primes, and $t \neq 0$). Its elements must all be *nilpotent*, due to:

Lemma 2.2. *If A is a noetherian ring then the ideal of nilpotent elements is precisely the intersection of the prime ideals.*

Proof. For any prime ideal P of A and any nilpotent $a \in A$ we have $a^n = 0 \in P$ for some $n > 0$, so $a \in P$ by primality. Hence, nilpotent elements lie in all primes. To show conversely that any $a \in A$ lying in every prime of A must be nilpotent, we prove the contrapositive: if a is *not* nilpotent then we will construct a prime ideal not containing a. The non-nilpotence of a implies that the localization A_a is *non-zero* (see HW2, Exercise 3). But A_a is *noetherian* since any ideal J of A_a is generated by the ideal $I \subset A$ of "numerators" of elements of J (so the finite generation of I implies the same for J). We know that every nonzero noetherian ring contains prime ideals (e.g., maximal ideals!), so A_a contains a prime ideal Q. Its preimage P under $A \to A_a$ is a prime ideal of A (since A/P is a subring of the ring A_a/Q that is a domain), and $a \notin P$ since the prime Q of A_a contains no units. ■

Since the prime ideal P of R/tR consists entirely of nilpotent elements and it is *finitely generated*, some power P^n vanishes. (The specific value of n is determined by the number of generators of P and a common exponent of nilpotence for each of them, in accordance with the multi-nomial theorem.) But $P = \mathfrak{m}/tR$, so $\mathfrak{m}^{n-1}P = 0$. By the hypothesis $tR \neq \mathfrak{m}$ we have $P \neq 0$, so $n > 1$. In other words, the descending chain of ideals $P, \mathfrak{m}P, \mathfrak{m}^2P, \dots, \mathfrak{m}^{n-1}P$ begins at a nonzero ideal and ends at the zero ideal, so there must be an integer $j \geq 0$ such that $\mathfrak{m}^jP \neq 0$ but $\mathfrak{m}^{j+1}P = 0$. Hence, any nonzero element of $\mathfrak{m}^jP$ is a nonzero element of $P = \mathfrak{m}/tR$ that is killed by $\mathfrak{m}$. Thus, by choosing such a nonzero element and picking a representative for it in $\mathfrak{m}$, we obtain an element $r \in \mathfrak{m}$ such that $r \notin tR$ but $\mathfrak{m}r \subseteq tR$.

Consider the ratio $a := r/t \in K$. By the choice of r we have $a \notin R$. We will show that a is nonetheless integral over R, so that will complete the proof. By the choice of r we also have $a\mathfrak{m} \subseteq R$. Thus, $J := a\mathfrak{m}$ is an ideal of R, since R-submodules of R are just ideals by another name. Since we chose $t \notin \mathfrak{m}^2$, certainly tR is not contained in $\mathfrak{m}^2$. But $r\mathfrak{m} \subseteq \mathfrak{m}^2$, so $tR \neq r\mathfrak{m}$. This says $J \neq R$, as desired. Since J is a proper ideal in the local noetherian ring R, it must be contained in the *unique* maximal ideal $\mathfrak{m}$, so

$$a \cdot \mathfrak{m} \subseteq \mathfrak{m}.$$

It now makes sense to consider the multiplication-by-a endomorphism of $\mathfrak{m}$. This is an R-linear endomorphism $T : \mathfrak{m} \to \mathfrak{m}$. The R-module $\mathfrak{m}$ is *finitely generated* since R is noetherian, and in our study of the basics of integrality we proved a "generalized Cayley-Hamilton theorem" to the effect that every linear endomorphism of a finitely generated module over a commutative ring satisfies a monic polynomial over the coefficient ring (with this polynomial constructed as a determinant of a suitable matrix over the coefficient ring). Thus, there is a monic polynomial $f \in R[x]$ such that $f(T) = 0$ as an endomorphism of $\mathfrak{m}$. But for any $b \in \mathfrak{m}$ we have $f(T)(b) = f(a)b$ by the definition of the endomorphism T, so $f(a)$ kills $\mathfrak{m}$. Since R is a *domain* and $\mathfrak{m} \neq 0$, this forces $f(a) = 0$. Thus, a is integral over R, as desired. ■