CHAPTER VIII

Transcendental Extensions

Both for their own sake and for applications to the case of finite extensions of the rational numbers, one is led to deal with ground fields which are function fields, i.e. finitely generated over some field k, possibly by elements which are not algebraic. This chapter gives some basic properties of such fields.

§1. TRANSCENDENCE BASES

Let K be an extension field of a field k. Let S be a subset of K. We recall that S (or the elements of S) is said to be algebraically independent over k, if whenever we have a relation

$$0 = \sum a_{(\nu)} M_{(\nu)}(S) = \sum a_{(\nu)} \prod_{x \in S} x^{\nu(x)}$$

with coefficients $a_{(\nu)} \in k$, almost all $a_{(\nu)} = 0$, then we must necessarily have all $a_{(\nu)} = 0$.

We can introduce an ordering among algebraically independent subsets of K, by ascending inclusion. These subsets are obviously inductively ordered, and thus there exist maximal elements. If S is a subset of K which is algebraically independent over k, and if the cardinality of S is greatest among all such subsets, then we call this cardinality the **transcendence degree or dimension** of K over k. Actually, we shall need to distinguish only between finite transcendence degree or infinite transcendence degree. We observe that the notion of transcendence degree bears to the notion of algebraic independence the same relation as the notion of dimension bears to the notion of linear independence.

We frequently deal with families of elements of K, say a family $\{x_i\}_{i \in I}$, and say that such a family is algebraically independent over k if its elements are distinct (in other words, $x_i \neq x_j$ if $i \neq j$) and if the set consisting of the elements in this family is algebraically independent over k.

A subset S of K which is algebraically independent over k and is maximal with respect to the inclusion ordering will be called a **transcendence base** of K over k. From the maximality, it is clear that if S is a transcendence base of K over k, then K is algebraic over $k(S)$.

Theorem 1.1. *Let K be an extension of a field k. Any two transcendence bases of K over k have the same cardinality. If Γ is a set of generators of K over k (i.e. $K = k(\Gamma)$) and S is a subset of Γ which is algebraically independent over k, then there exists a transcendence base $\mathcal{B}$ of K over k such that $S \subset \mathcal{B} \subset \Gamma$.*

Proof. We shall prove that if there exists one finite transcendence base, say $\{x_1, \ldots, x_m\}$, $m \geqq 1$, then any other transcendence base must also have m elements. For this it will suffice to prove: If $w_1, \ldots, w_n$ are elements of K which are algebraically independent over k then $n \leqq m$ (for we can then use symmetry). By assumption, there exists a non-zero polynomial f_1 in $m + 1$ variables with coefficients in k such that

$$f_1(w_1, x_1, \ldots, x_m) = 0.$$

Furthermore, by hypothesis, w_1 occurs in f_1, and some x_i also occurs in f_1, say x_1. Then x_1 is algebraic over $k(w_1, x_2, \ldots, x_m)$. Suppose inductively that after a suitable renumbering of $x_2, \ldots, x_m$ we have found $w_1, \ldots, w_r$ $(r < n)$ such that K is algebraic over

$$k(w_1, \ldots, w_r, x_{r+1}, \ldots, x_m).$$

Then there exists a non-zero polynomial f in $m + 1$ variables with coefficients in k such that

$$f(w_{r+1}, w_1, \ldots, w_r, x_{r+1}, \ldots, x_m) = 0,$$

and such that w_{r+1} actually occurs in f. Since the w's are algebraically independent over k, it follows that some x_j $(j = r + 1, \ldots, m)$ also occurs in f. After renumbering we may assume $j = r + 1$. Then x_{r+1} is algebraic over

$$k(w_1, \ldots, w_{r+1}, x_{r+2}, \ldots, x_m).$$

Since a tower of algebraic extensions is algebraic, it follows that K is algebraic over $k(w_1, \ldots, w_{r+1}, x_{r+2}, \ldots, x_m)$. We can repeat the procedure, and if $n \geqq m$ we can replace all the x's by w's, to see that K is algebraic over $k(w_1, \ldots, w_m)$. This shows that $n \geqq m$ implies $n = m$, as desired.