Math 145. Homework 1

"As long as Algebra and Geometry were separated, their progress was slow and their use limited; but once these sciences were united, they lent each other mutual support and advanced rapidly together towards perfection." Lagrange (1795)

Ch 1: 1.4, 1.5, 1.10, 1.17, 1.22

1. Find all solutions to $x^2 + 2y^2 = 3$ with $x, y \in \mathbf{Q}$, and prove that $x^2 + 3y^2 = 2$ has no solutions in $\mathbf{Q}$. Can you state a generalization for $ax^2 + by^2 = c$ with $a, b, c \in k^\times$ and k any field not of characteristic 2? Draw pictures.

2. Let k be an algebraically closed field. Thinking about tangency, give an example of affine algebraic sets Z_1, Z_2 in k^2 with $\underline{I}(Z_1 \cap Z_2) \neq \underline{I}(Z_1) + \underline{I}(Z_2)$. Draw a picture.

3. This exercise develops basic facts for manipulating polynomials in several variables.

(*i*) Let R be a ring. Define $R[X_1, \dots, X_n]$ in terms of 'sequences of coefficients', define on it a structure of commutative R-algebra, and prove that it has the following universal mapping property: for any R-algebra A and any $a_1, \dots, a_n \in A$, there is a unique map of R-algebras $R[X_1, \dots, X_n] \to A$ which sends X_i to a_i. The image of f under this map is called the *value* of f at $(a_1, \dots, a_n)$. Note that when $R = 0$, the only R-algebra is R itself (e.g., $R[X] = R$ for $R = 0$).

(*ii*) If I is the ideal in $R[X_1, \dots, X_n]$ generated by elements f_α, then state and prove a universal mapping property for the R-algebra $R[X_1, \dots, X_n]/I$. Interpret this in the special case $I = (X_1 - r_1, \dots, X_n - r_n)$ for $r_j \in R$. Conclude that $R[X]$ is not isomorphic to R as an R-algebra if $R \neq 0$, but give an example of a non-zero ring R for which there is an isomorphism $R[X] \simeq R$ as abstract rings.

(*iii*) For $f \in R[X]$, $g \in R[Y]$, prove that there are unique isomorphisms of R-algebras

$$(R[Y]/(g))[X]/(f) \simeq R[X, Y]/(f, g) \simeq (R[X]/(f))[Y]/(g)$$

determined by "$X \mapsto X$" and "$Y \mapsto Y$". Generalize for any finite number of variables, with (f) and (g) replaced by any ideals in the corresponding polynomial rings.

4. (*i*) If A is a UFD, prove that $A[X_1, \dots, X_n]$ is a UFD (e.g., $A = \mathbf{Z}$ or A a field). Prove rigorously that $k[X, Y, Z, W]/(XY - ZW)$ is a domain but is not a UFD, where k is an algebraically closed field.

(*ii*) Prove that if k is a field and $f \in k[X]$ with positive degree is a product of distinct irreducible polynomials, then $Y^2 - f \in k[X, Y]$ is irreducible. (Hint: $k[X, Y] = (k[X])[Y] \subset k(X)[Y]$.)