Math 145. Homework 2

1. (i) For a field K, prove that any domain A finite-dimensional over K is a field.

(ii) If R is a non-zero ring and $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ are distinct maximal ideals in R, prove that the natural map $R/\cap \mathfrak{m}_i \to \prod(R/\mathfrak{m}_i)$ is an isomorphism (hint: show that $(1, 0, \dots, 0)$ is hit).

2. Let k be an algebraically closed field, and $S \subseteq k^n$ a subset, so $\underline{Z}(\underline{I}(S))$ is an affine algebraic set in k^n that contains S. Prove that any affine algebraic set $Z = \underline{Z}(J)$ in k^n that contains S must contain $\underline{Z}(\underline{I}(S))$ (hint: apply $\underline{I}$ to the inclusion $S \subseteq \underline{Z}(J)$ and use the Nullstellensatz). Explain why this says exactly the following: $\underline{Z}(\underline{I}(S))$ is the *closure* of S relative to the Zariski topology on k^n.

3. Let A be a ring. For $a \in A$, define $A_a = A[X]/(1 - aX)$. This is naturally an A-algebra, and is sometimes denoted $A[1/a]$ (but we do not require A to be a domain, so the fraction notation is merely suggestive).

(i) Note that a is a unit in A (i.e., admits a multiplicative inverse). Formulate a universal mapping property for the A-algebra A_a, and prove that $b \in A$ maps to 0 in A_a if and only if $a^n b = 0$ in A for some positive integer n. Conclude that $A_a \neq 0$ if and only if a is not nilpotent. Also show that if A is a domain with fraction field K, then there is a canonical map $A_a \to K$ which is *injective* (so A_a is a domain) and describe the image.

(ii) For every $f \in A_a$, show that for some large integer n, $a^n f \in A_a$ is in the image of the natural map $A \to A_a$. Conclude that if A has no non-zero nilpotents, then A_a has the same property. When $A = k[X, Y]/(XY)$ and $a = Y$ with k a field, show that $A_a \simeq k[Y, Y^{-1}] = k[Y]_Y$ as k-algebras and that the (non-injective!) natural map $A \to A_a$ induces an injective map of sets

$$\mathrm{Hom}_{k-\mathrm{alg}}(A_a, k) \to \mathrm{Hom}_{k-\mathrm{alg}}(A, k);$$

interpret this map geometrically when k is algebraically closed.

(iii) Let I be an ideal in A. Use the universal property in (i) to define a natural map $A_a \to (A/I)_a$ and prove that this induces an isomorphism $A_a/(I \cdot A_a) \simeq (A/I)_a$ (hint: use mapping properties to construct the inverse map by pure thought).

4. Let X be a topological space. We say that X is *quasi-compact* if every open covering has a finite subcovering (this terminology simply emphasizes that we do not assume the space to be Hausdorff). We say X is *noetherian* if every decreasing sequence $Z_1 \supseteq Z_2 \supseteq \dots$ of closed sets in X terminates (i.e., $Z_n = Z_{n+1}$ for all large n).

(i) Prove that k^n with its Zariski topology is noetherian for any algebraically closed field k (hint: Nullstellensatz). Also show that a closed set in a noetherian topological space is noetherian, so every affine algebraic set in k^n is noetherian for the subspace topology.

(ii) Show that if X is noetherian, then it is quasi-compact and *all* subspaces are noetherian. Conversely, if all open subsets in X are quasi-compact, then show that X is noetherian.

(iii) If $X \to Y$ is a surjective continuous map and X is noetherian then so is Y.

5. Let $n, m \geq 1$ and let $M_{m,n} = k^{mn} = \{(x_{ij})\}$ denote the 'affine space' of $m \times n$ matrices over an algebraically closed field k. Prove that for any r, the subset corresponding to matrices with rank $\leq r$ is an affine algebraic set in this k^{mn} (consider determinants of many minors). When $m = n$, prove that the subset corresponding to invertible matrices is Zariski-open.