Math 145. Homework 4

2.8, 2.11 (and find the coordinate ring of $G(f)$), 2.12 (draw a picture), 2.13, 2.17 in the book. As usual, k is algebraically closed (and "isomorphism" is via coordinate rings; see Exercise 2 below).

0. Prove UFD's are integrally closed, and that $k[X,Y]/(Y^2 - X^3)$ is a domain but not integrally closed. If $\mathrm{char}(k) \neq 2$, prove $k[x,y,z]/(xy - z^2)$ is an integrally closed domain but not a UFD (hint: this is $A[z]/(z^2 - a)$ for the UFD $A = k[x,y]$ and $a = xy$); extra credit for handling $\mathrm{char}(k) = 2$.

1. (*i*) Let $Z = \underline{Z}(I) \subseteq k^n$ be an affine algebraic set, with I an ideal in $k[X_1, \ldots, X_n]$; don't assume I is radical. For any $z \in Z$ and open U in Z around z, prove that there exists $f \in k[X_1, \ldots, X_n]/I$ such that the open set $Z_f \stackrel{\text{def}}{=} \{z \in Z \mid f(z) \neq 0\}$ contains z and lies inside U. The upshot of this result is that the 'basic opens' Z_f form a base of opens for the Zariski topology on Z.

(*ii*) For $n, m \geq 1$, prove that the natural bijection $k^n \times k^m \to k^{n+m}$ is *not* continuous for the Zariski topology on the right and the product topology from the Zariski topologies on the left.

2. Let $Z \subseteq k^n$, $Z' \subseteq k^{n'}$ be two affine algebraic sets and let $f : Z \to Z'$ be a "polynomial map" between them, with $f^* : k[Z'] \to k[Z]$ the corresponding map of k-algebras.

(*i*) If $Z'' \subseteq k^{n''}$ is a third affine algebraic set and $f' : Z' \to Z''$ is a polynomial map, show that $f' \circ f : Z \to Z''$ is a polynomial map and $(f' \circ f)^* = f^* \circ f'^*$. Also show that if $n' = n$ and $Z' = Z$ inside of $k^{n'} = k^n$, then f is the identity if and only if f^* is the identity (so the "dictionary" between affine algebraic sets and finitely generated reduced k-algebras is compatible with maps).

(*ii*) Recall from class that for an affine algebraic set $\underline{Z}(I') \subseteq Z'$ with I' an ideal in $k[Z']$, we have

$$f^{-1}(\underline{Z}(I')) = \underline{Z}(f^*(I')k[Z]),$$

though the ideal $f^*(I')k[Z]$ might not be radical, even if I is radical. Describe this for the projection of $\{(x,y) \in k^2 \mid x = y^2\}$ to the x-axis, with I' corresponding to the maximal ideal of a point on the x-axis. When is $k[Z]/f^*(I')k[Z]$ non-reduced? Draw pictures, and don't ignore characteristic 2.

3. Let k be an algebraically closed field. Let A be the image of the map $k[T_1, T_2, T_3, T_4] \to k[X,Y]$ which sends $T_1 \mapsto X$, $T_2 \mapsto XY$, $T_3 \mapsto Y^2$, $T_4 \mapsto Y^3$, so A is a finitely generated domain over k. Let P be the kernel and $Z := \underline{Z}(P) \subseteq k^4$ an irreducible algebraic set, and t_j the image of T_j in A.

(*i*) Show that A is the k-subalgebra of $k[X,Y]$ consisting of elements of the form $c + f$ for $c \in k$ and $f \in (X, Y^2)$, and check that $t_2^2 = t_1^2 t_3, t_2 t_4 = t_1 t_3^2, t_2 t_3 = t_1 t_4, t_3^3 = t_4^2$. Show also that the inclusion $A \to k[X,Y]$ is a finite map, identifying the fraction fields of these domains. Let $\pi : k^2 \to Z \subset k^4$ be the corresponding geometric map ($(x,y) \mapsto (x, xy, y^2, y^3)$). Show that for $a \in A$ with $a \notin (X,Y)$, necessarily $Y \notin A_a$ inside of this common fraction field.

(*ii*) By finiteness, π is surjective onto Z and *closed*. Show that $\pi^{-1}(0,0,0,0) = (0,0)$, the two open sets defined by $t_1 \neq 0$, $t_3 \neq 0$ cover $Z - \{(0,0,0,0)\}$, and the maps $A_{t_1} \to k[X,Y]_X$ and $A_{t_3} \to k[X,Y]_Y$ are isomorphisms. (Geometrically, view Z as obtained from k^2 by crunching up $(0,0) \in k^2$ into a nasty singularity, with $k^2 \to Z$ inducing an "isomorphism" away from the origin.)

(*iii*) Show that $f = t_2/t_1 \in k(Z)$ is a rational function defined on $Z - \{(0,0,0,0)\}$ and *prove* it is *not* defined at $(0,0,0,0)$ (hint: use the map π). This gives an example an irreducible surface with a rational function whose pole set is a single point! By contrast, recall from the previous homework that if $Z \subseteq k^n$ is an irreducible affine variety for which $k[Z]$ is a UFD (e.g., $Z = k^n$), then the pole set of any $f \in k(Z)$ is a finite (possibly empty) union of "codimension 1" subvarieties.

4. Let $Z \subset \mathbf{C}^n$ and $Z' \subset \mathbf{C}^m$ be closed in the classical topology, and $f : Z \to Z'$ a continuous map with *finite* fibers. Prove that f is closed if and only if any sequence $\{z_n\}$ in Z with $f(z_n) \to z' \in Z'$ has a convergent subsequence. (Hint: bounded sequences in $\mathbf{C}^n$ have convergent subsequences)