Math 145. Homework 5

1. A *multiplicative set* S in a domain R is a subset of $R-\{0\}$ containing 1 and stable under multiplication. Define $S^{-1}R \subset \mathrm{Frac}(R)$ to be the set of fractions r/s with $s \in S$.

(*i*) Prove $S^{-1}R$ is a subring of $\mathrm{Frac}(R)$, and that we can use $S = R - P$ for a *prime* ideal P of R (in which case $S^{-1}R$ is denoted as R_P). Also show that every ideal in $S^{-1}R$ has the form $S^{-1}J$ for an ideal J of R (hint: numerators), and deduce that $S^{-1}R$ is noetherian when R is. Finally, show that $P \mapsto P \cap R$ and $P' \mapsto P \cdot S^{-1}R$ are inverse bijections between the sets of primes of $S^{-1}R$ and primes of R *disjoint* from S.

(*ii*) If $A \hookrightarrow B$ is an injection of domains with B a finitely generated A-algebra, and $S \subseteq A - \{0\}$ is a multiplicative set such that $S^{-1}B$ is finite (resp. finite free) as an $S^{-1}A$-module, prove $A_s \to B_s$ is finite (resp. finite free) for some $s \in S$.

(*iii*) If Z is a non-empty affine algebraic set, $g \in k[Z]$, prove Z_g is dense in Z if and only if g is not a zero divisor in $k[Z]$ (i.e., $gf = 0 \Rightarrow f = 0$). Give an example where Z_g is non-empty and not dense in Z.

(*iv*) If Z is an affine variety of dimension d and $\varphi : k[Z] \to A = k[X_1, \dots, X_m]/I$ makes A a finite *free* $k[Z]$-module but I is *not* assumed to be radical, prove that *all* irreducible components of $\underline{Z}(I)$ are of dimension d. Give a counterexample with radical I if φ is merely assumed to be injective (think geometrically).

2. Let $Z \subseteq k^n$, $Z' \subseteq k^m$ be affine varieties with the same dimension, $A = k[Z]$, $A' = k[Z']$. Let $f : Z \to Z'$ be a polynomial map between them, with dense image. This exercise interprets "field degree" geometrically.

(*i*) Show that the naturally associated map of function fields $k(Z') \hookrightarrow k(Z)$, compatible with $k[Z'] \to k[Z]$, is a *finite* extension of fields, say of degree d. We call d the *degree* of f. Give an example with $d = 3$.

(*ii*) Prove that the subring $A[1/a' \,|a' \in A' - \{0\}]$ of $k(Z)$ consisting of elements with denominator from $A' - \{0\}$ is a domain finite over the field $k(Z')$, and so is equal to $k(Z)$. Conclude the existence of some non-zero $a' \in A'$ such that $A'_{a'} \to A_{a'}$ makes $A_{a'}$ a finite free $A'_{a'}$-module of rank d, and that the induced map of topological spaces $f^{-1}(Z'_{a'}) \to Z'_{a'}$ has a finite *non-empty* fiber of size at most d over each point.

(*iii*) For any $a \in A - \{0\}$ and monic $h \in A_a[T]$, prove that the natural map of rings $A_a[T]/(h) \to k(Z)[T]/(h)$ is injective. Using this and the primitive element theorem from field theory, prove that if $k(Z)$ is separable over $k(Z')$ (e.g., if k has characteristic 0), then a' in (*ii*) can be chosen so that there is an isomorphism of $A'_{a'}$-algebras $A_{a'} \simeq A'_{a'}[T]/(h)$, where $h \in A'_{a'}[T]$ is a monic polynomial for which the derivative $h'(T)$ has invertible (i.e., unit) image in $A'_{a'}[T]/(h)$. (hint: chase denominators)

(*iv*) Assuming $k(Z)$ to be separable over $k(Z')$, for a' as in the final part of (*iii*) prove that the map of open sets $Z_{a'} \to Z'_{a'}$ is surjective, with *exactly* d points in each fiber. This is the geometric interpretation of the degree of a map. Give a counterexample if the separability hypothesis is dropped.

(*v*) For an explicit example of the preceding phenomonena, consider $Z = k^2 \to k^2 = Z'$ defined by $(x, y) \mapsto (xy, y)$. Prove that this induces an isomorphism on function fields and find a non-zero $a' \in k[Z'] = k[X, Y]$ for which the preimage of every point in $Z'_{a'}$ consists of a single point in Z.

3. Prove that if Z is an affine variety of dimension d and $k[Z]$ is a UFD, the irreducible affine subvarieties in Z with dimension $d-1$ are exactly those closed sets of the form $\underline{Z}(f)$ for an irreducible $f \in k[Z]$.

4. This exercise proves a crucial result linking function fields and coordinate rings (see (iv) below). Let A be a domain with fraction field K. Let $\widetilde{A}$ denote the integral closure of A in K.

(*i*) For any non-zero $a \in A$, prove that $(\widetilde{A})_a$ is integrally closed and is the integral closure of A_a in K.

(*ii*) If $f_1, \dots, f_n \in A$ generate 1, prove that $\cap A_{f_i} = A$, the intersection inside K. (Hint: for an element in the intersection, define an 'ideal of denominators' in A and show it contains some power of each f_i.) Either assuming A to be noetherian or granting that proper ideals in a ring always lie inside of a maximal ideal (by Zorn), prove also that $\cap A_{\mathfrak{m}} = A$, where the intersection inside K is taken over all maximal ideals of A.

(*iii*) Assuming A to be noetherian (or granting that all proper ideals in a ring lie inside of a maximal ideal), prove that A is integrally closed if and only if $A_{\mathfrak{m}}$ is integrally closed for all maximal ideals $\mathfrak{m}$ of A. Likewise, without any noetherian hypotheses or use of Zorn's Lemma, for $\{f_j\}$ as in (*ii*) show that A is integrally closed if and only if A_{f_j} is integrally closed for all j.

(*iv*) If $Z \subseteq k^n$ is an affine *variety* (i.e., irreducible) and $f \in k(Z)$ has empty pole set, prove rigorously using (ii) that $f \in k[Z]$. Note that this is *not* just a matter of chasing definitions!