Math 145. Homework 7

Do 2.35 ("form" means "homogenous polynomial"; a polynomial whose non-zero monomial terms have the same degree), 2.44 (let J be radical), 3.2(a)–(c), 3.12 (for 3.2 and 3.12 assume characteristic 0), 3.17 (just compute the intersection numbers $I(C, D)$, $I(C, E)$) in the text.

1. (i) For any $v = (c_1, \dots, c_n) \in k^n$ and $f \in k[x_1, \dots, x_n]$, define the *directional derivative* $\partial_v f \in k[x_1, \dots, x_n]$ to be $\sum c_i \partial_{x_i} f$. Show that this is coordinate-independent in the sense that for any linear automorphism L of k^n, $(\partial_v f) \circ L = \partial_{L^{-1}(v)}(f \circ L)$ as functions on k^n.

(ii) Show that for any $\xi \in k^n$ and $v \in k^n$ the value $(\partial_v f)(\xi)$ for $f \in k[x_1, \dots, x_n]$ only depends on $f \bmod \mathfrak{m}_\xi^2$. (Hint: use (i) to reduce to the case $\xi = (0, \dots, 0)$.) Deduce that the map $\alpha_\xi : k^n \to \mathrm{T}_\xi(k^n) = (\mathfrak{m}_\xi/\mathfrak{m}_\xi^2)^*$ defined by $v \mapsto (f \mapsto (\partial_v f)(\xi))$ is a k-linear isomorphism, and that if $\tau : k^n \to k^n$ is translation by v_0 then $\mathrm{d}\tau(\xi) : \mathrm{T}_\xi(k^n) \simeq \mathrm{T}_{\xi+\tau_0}(k^n)$ carries α_ξ to $\alpha_{\xi+v_0}$. In this sense, we have a "translation-invariant" canonical identification of the k-vector space k^n with the tangent space at any of its points.

(iii) For a prime ideal $P = (h_1, \dots, h_r)$ in $k[x_1, \dots, x_n]$ and $Z = \underline{Z}(P)$, consider the inclusion map $\iota : Z \to k^n$. Show that for any $z \in Z$, $\mathrm{d}\iota(z) : \mathrm{T}_z(Z) \to \mathrm{T}_z(k^n) = k^n$ (using (ii) for the final equality) is an injection that identifies $\mathrm{T}_z(Z)$ with the kernel of the Jacobian matrix $((\partial_{x_j} h_i)(z))$ at z. (Hint: use (ii) to reduce to the case $z = (0, \dots, 0)$.)

2. Let $V \subseteq k^n$ and $V' \subseteq k^m$ be irreducible closed subsets, and let $\varphi : k(V) \to k(V')$ be an injective k-algebra map between the function fields.

(i) Find nonzero $a' \in k[V']$ such that φ carries $k[V]$ into $k[V']_{a'}$. Giving $V'_{a'}$ its natural structure of affine variety via the Rabinowitz trick (see HW6, Exercise 4(ii)), deduce that φ is induced by a dominant map of affine varieties $V'_{a'} \to V$.

(ii) If φ is an *isomorphism*, find nonzero $a' \in k[V']$ and $a \in k[V]$ such that φ arises from an isomorphism of k-algebras $k[V]_a \simeq k[V']_{a'}$ (i.e., φ arises from an isomorphism $V'_{a'} \simeq V_a$ when using the natural affine variety structures on these loci). In other words, the function field of an affine variety "remembers" information up to discarding a proper closed subset.

3. It is a general fact that if F'/F is a finitely generated extension of fields and F is *perfect* (e.g., algebraically closed), then F' admits a transcendence basis $\{x_i\}$ that is *separating* in the sense that the finite extension $F'/F(x_1, \dots, x_n)$ is *separable*. In particular, by the primitive element theorem this is primitive. The case of interest to us is $F = k$.

(i) Let $Z \subset k^n$ be an irreducible closed subset of dimension d. Choose a separating transcendence basis $\{x_1, \dots, x_d\}$ of $k(Z)$ over k. Show that $k(Z) \simeq k(x_1, \dots, x_d)[T]/(h)$ over k for an irreducible $h \in k[x_1, \dots, x_d, T]$ that is monic in T and has positive T-degree. (Hint: clear denominators appropriately.)

(ii) Using Exercise 2(ii), deduce that Z and $V = \underline{Z}(h) \subset k^{d+1}$ have isomorphic dense open subsets in the sense that there are basic affine open subsets of Z and V whose coordinate rings are identified under the equality $k(Z) = k(V)$ defined by Exercise 2(ii).

(iii) For any nonzero $a \in Z$ and point $z \in Z_a$, explain how to compute $\mathscr{O}_{Z,z}$ in terms of $k[Z]_a$. (Hint: identify Z_a with the set of maximal ideals of $k[Z]_a$.) By applying the same to V, and using the result from class that the locus of smooth points in V is a non-empty open subset, prove that the locus of smooth points in Z contains a non-empty open subset of Z.