Math 145. Homework 9

1. Prove $\mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^{(n+1)(m+1)-1}$ via $S([x_i],[y_j]) = [x_0y_0, \dots, x_iy_j, \dots, x_ny_m]$ is a well-defined morphism.

2. If A is a finitely generated, reduced k-algebra and $(X, \mathscr{O}_X)$ is an abstract algebraic set, explain how a morphism $X \to \mathrm{MaxSpec}(A)$ gives rise to a map of k-algebras $A \to \mathscr{O}_X(X)$. Working locally on X, show that the map $\mathrm{Hom}(X, \mathrm{MaxSpec}(A)) \to \mathrm{Hom}_{k\text{-alg}}(A, \mathscr{O}_X(X))$ is bijective (hint: begin with the case of affine X, and then work locally on a general X, covering an overlap of open affines with open affines).

3. An ideal $I \subset k[x_0, \dots, x_n]$ is *homogenous* if for every $f \in I$ each homogeneous part f_d of f lies in I.

(i) Prove that any homogeneous ideal is generated by finitely many homogenous elements, and that conversely the ideal generated by any finite set of homogenous elements is in fact a homogenous ideal.

(ii) For a homogenous ideal J, show that the zero locus of J in $\mathbf{A}_k^{n+1} - \{0\}$ is the full preimage of the common zero locus $\underline{Z}(J) \subset \mathbf{P}^n$ of the *homogenous* elements of J.

(iii) For a non-empty closed set $V \subset \mathbf{P}^n$ and its closed preimage $V' \subset \mathbf{A}^{n+1} - \{0\}$, prove the closure $\overline{V}'$ equals $V' \cup \{0\}$ and that $\underline{I}(\overline{V}')$ is homogenous. (hint: If $f = \sum f_d \in \underline{I}(\overline{V}')$ with homogenous parts f_d then show $f(tx_0, \dots, tx_n) = \sum t^d f_d \in \underline{I}(\overline{V}')$ for any $t \in k^\times$. Then apply van der Monde using enough t's.)

(iv) Let $Z \subset \mathbf{A}^n$ be non-empty closed, and $I = \underline{I}(Z) \subset k[t_0, \dots, t_{n-1}]$. Identify $\mathbf{A}^n$ with the standard affine open $U_n = \{x_n \neq 0\} \subset \mathbf{P}^n$ as usual. For $f \in k[t_0, \dots, t_{n-1}]$ of degree $d > 0$, defines its x_n-*homogenization* $\widetilde{f}(x_0, \dots, x_n) = x_n^d f(x_0/x_n, \dots, x_{n-1}/x_n)$. Show that the homogenous ideal $\widetilde{I}$ generated by the $\widetilde{f}$ for $f \in I$ is radical (hint: show a homogenous h lies in $\widetilde{I}$ if and only if its x_n-dehomogenization vanishes on Z) and that $\underline{Z}(\widetilde{I}) \subset \mathbf{P}^n$ is the closure of Z (hint: apply (iii) to a closed set $V \subset \mathbf{P}^n$ containing Z).

4. (i) If X and X' are irreducible abstract algebraic sets then prove $X \times X'$ is irreducible. (hint: reduce to affine X and X', and if $Z \subset X \times X'$ is a proper closed subset then prove that $U_Z := \{x \in X \mid Z \not\supset \{x\} \times X'\}$ is a non-empty *open* subset of X by first showing $U_Z = \cup_i U_{\underline{Z}(f_i)}$ for generators $f_1, \dots, f_n$ of $\underline{I}(Z)$.)

(ii) If V is an irreducible abstract algebraic set covered by finitely many affine opens, prove that *every* non-empty affine open subset of V has the same dimension, say d. Then show that every non-empty open subset of V is noetherian with dimension d. In the setting of (i), deduce that if X and X' are covered by finitely many affine opens then so is $X \times X'$ and $\dim(X \times X') = \dim(X) + \dim(X')$.

5. Let X be an irreducible abstract algebraic set.

(i) For any non-empty affine open $U, U' \subset X$ and a non-empty affine open $V \subset U \cap U'$, show that the composite isomorphism $\phi_{U',U} : k(U) \simeq k(V) \simeq k(U')$ is independent of V, and that $\phi_{U'',U'} \circ \phi_{U',U} = \phi_{U'',U}$ as isomorphisms $k(U) \simeq k(U'')$. In this way, the function fields of all non-empty affine opens in X are *canonically* identified; we call this field the *function field* of X and denote it $k(X)$.

(ii) For the affine opens $U_i \subset \mathbf{P}^n$, show that $t_{ji} \mapsto x_j/x_i$ carries all $k(U_i)$ onto the subfield of $k(x_0, \dots, x_n)$ whose nonzero elements are the ratios f/g where $f, g \in k[x_0, \dots, x_n]$ are homogeneous of the same degree, *compatibly with* the isomorphisms $k(U_i) \simeq k(U_j)$ in (i). This "computes" $k(\mathbf{P}^n)$.

(iii) Now let X and Y be irreducible and separated. A rational map (U, f) from X to Y is *dominant* if $f(U)$ is dense in Y. Show that this denseness condition is independent of the representative (U, f) within the equivalence class, and explain how to define *composition* of dominant rational maps so that it is well-defined and *associative*. Show that there is a unique way to associate to any dominant rational map from X to Y a k-algebra map of fields $f^* : k(Y) \to k(X)$ so that (1) $(f \circ g)^* = g^* \circ f^*$ for all dominant rational g from W to X and (2) f^* is the obvious map of fraction fields when f is a dominant morphism between affines.

(iv) Conversely, for irreducible and separated X and Y, prove that any k-algebra map $k(Y) \to k(X)$ has the form f^* for a unique dominant rational map f from X to Y (hint: chase denominators relative to a choice of presentation of the coordinate ring of a non-empty affine open in Y as a quotient of a polynomial ring over k). This gives a useful "geometric interpretation" of maps between function fields.

Deduce that $k(X) \simeq k(Y)$ over k if and only if there exist non-empty opens $U \subseteq X$, $V \subseteq Y$ and an isomorphism $U \simeq V$; we say X and Y are *birationally isomorphic*. Find an isomorphism between function fields of $\{y^2 = x^2(x-1)\}$ and $\{v^2 = u^3\}$ (hint: identify each with $k(t)$) and an explicit isomorphism between explicit non-empty open affines inducing your function field isomorphism. Illustrate with pictures.