Math 210B. Artinian rings and modules

1. Basic formalism

Let R be a commutative ring. An R-module M is called *artinian* if it satisfies the descending chain condition (dcc) for submodules: any descending sequence $M_1 \supset M_2 \supset \dots$ stabilizes (i.e., $M_n = M_{n+1}$ for sufficiently large n, or equivalently for some large n_0 we have $M_n = M_{n_0}$ for all $n \geq n_0$).

Example 1.1. Suppose R is finite-dimensional over a field k and M is a finitely generated R-module. Clearly M is finite-dimensional as a k-vector space. Thus, any descending chain of R-submodules of M is also a descending chain of k-subspaces. Any such chain in a *finite-dimensional* k-vector space must stabilize because the dimensions of the members of the chain constitute a monotonically decreasing sequence of non-negative integers, which of course must eventually stabilize, so M is artinian.

Example 1.2. A vector space V over a field k is artinian as a k-module if and only if it is finite-dimensional over k (in which case it is clearly noetherian as a k-module too!). The implication "$\Leftarrow$" is clear for dimension reasons seen in the preceding example. For the converse, suppose V is not finite-dimensional and let $\{e_i\}_{i \in I}$ be a basis. By infinitude of the basis, we can choose an *infinite sequence* of pairwise distinct basis vectors $e_{i_1}, e_{i_2}, \dots$. Letting W_n be the span of e_{i_m}'s for $m \geq n$, clearly $W_1 \supset W_2 \supset \dots$ violates dcc for V.

Many basic features of artinian modules proceed similarly to the case of noetherian modules in both statements and proofs, as we will illustrate below. The real dichotomy emerges when we focus on the special case of artinian *rings*: those rings R that are artinian as modules over themself, or in other words satisfying dcc for ideals (reminiscent of the definition of a noetherian ring using acc).

Example 1.3. Very few noetherian rings are artinian. In fact, *any* domain D that is not a field cannot be artinian. Indeed, such a D admits a nonzero non-unit d, so the descending chain of ideals $(d) \supset (d^2) \supset (d^3) \supset \dots$ does not stabilize (why not?).

It will turn out that artinian rings are always noetherian, and of a very special type. The asymmetry between the two concepts "artinian" and "noetherian" in the ring will ultimately stem from the preceding example.

Here are some basic results that proceed identically to the noetherian case studied in 210A:

Lemma 1.4. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. Then M is artinian if and only if M' and M'' are artinian.*

Proof. If M is artinian then M' inherits dcc from M since all submodules of M' are also submodules of M, and M'' inherits dcc from M since any descending chain of submodules of M'' is the image of its preimage that is a descending chain in M.

Suppose instead that M' and M'' satisfy dcc and let $\{M_n\}$ be a descending chain of submodules in M. The intersections $M'_n = M' \cap M_n$ constitute a descending chain in M', so for large n this stabilizes. By dropping some initial terms and relabeling we can therefore assume that $M'_n \subset M'$ is the same for all n. Thus, $M_n/M'_1 \to M/M' = M''$ is injective for all n and as such constitutes a descending chain in M''. By dcc for M'' we therefore have stabilization for large n, so the inclusion $M_{n+1} \subset M_n$ for large n becomes an equality modulo M'_1 for large n. Hence, $M_{n+1} = M_n$ for large n as desired. ∎

Lemma 1.5. *Let $(0) = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_s = M$ be a finite chain of submodules of M. Then M is artinian if and only if each successive quotient M_n/M_{n-1} is artinian for $1 \leq n \leq s$.*

Proof. The case $s = 1$ has no content, and the case $s = 2$ is exactly Lemma 1.4. In general we induct on s: if $s > 2$ then we apply Lemma 1.4 to

$$0 \to M_1 \to M \to M/M_1 \to 0$$

and note that M/M_1 contains the chain $\{M_{n+1}/M_1\}_{0 \leq n \leq s-1}$. ■

Example 1.6. If $M_1, \ldots, M_n$ are R-modules, then $M_1 \oplus \cdots \oplus M_n$ is artinian if and only if each M_j is artinian.

Example 1.7. Let $M = \mu_{p^\infty}$ be the infinite group of p-power roots of unity in $\mathbf{C}$ for a prime p; this is a rising union of cyclic subgroups μ_{p^n} of order p^n (with μ_{p^n} the subgroup of p^nth roots of 1). We claim that M is artinian as a $\mathbf{Z}$-module. It suffices to check that the only *proper* $\mathbf{Z}$-submodules of M are the finite subgroups μ_{p^n}.

Every element of M has order p^n for some $n \geq 0$, and if $m \in M$ has order p^n then it is a primitive p^nth root of unity and so generates the cyclic group μ_{p^n} of order p^n. Thus, if $M' \subset M$ is a $\mathbf{Z}$-submodule and $m' \in M'$ has order p^n for some $n \geq 0$ then m' generates μ_{p^n} and so $M' \supset \mu_{p^n}$.

The crucial point is that the collection of subgroups $\{\mu_{p^n}\}$ is totally ordered by inclusion: $\mu_{p^n} \subset \mu_{p^{n'}}$ when $n' \geq n$, so if M' contains an element of order p^n then it contains μ_{p^r} for all $r \leq n$. If $M' \neq M$ then it does not contain some μ_{p^e} with $e \geq 1$ (as the union of all μ_{p^n}'s is equal to M), so M' cannot contain $\mu_{p^{n'}}$ for *any* $n' \geq e$ and hence all elements of M' must have p-power order p^r with $r < e$! This forces $M' \subset \mu_{p^{e-1}}$.

Here is the crucial example:

Example 1.8. Let A be a noetherian local ring whose maximal ideal $\mathfrak{m}$ is nilpotent, say $\mathfrak{m}^n = 0$ for some $n \geq 1$. We claim that A is artinian as an A-module (equivalently, its ideals satisfy dcc).

By nilpotence of $\mathfrak{m}$, we have a finite descending chain of ideals $\{\mathfrak{m}^j\}_{0 \leq j \leq n}$. By Lemma 1.5, A is artinian as an A-module if and only if each successive quotient $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ is artinian as an A-module. But this quotient as an A-module is even a module over the residue field $k := A/\mathfrak{m}$, so its A-submodules are the same as its k-subspaces. Thus, it is enough to show that such successive quotients are finite-dimensional as k-vector spaces. That is the same as being finitely generated as an A-module, which in turn follows from each $\mathfrak{m}^j$ being finitely generated as an ideal in A (as A is noetherian, by hypothesis).

2. Structure for rings

Now we take up the main task, which is to show that artinian commutative rings R are noetherian of a very special type. More specifically, we'll show that every nonzero artinian ring is a direct product of finitely many of the type in Example 1.8. We begin with:

Lemma 2.1. *There are no strict containments among prime ideals of R (so they are all maximal and minimal), and there are only finitely many prime ideals.*

Proof. For the first assertion, it suffices to show that all prime ideals are maximal. Let $\mathfrak{p} \subset R$ be a prime ideal, so the quotient $A := R/\mathfrak{p}$ is a domain that inherits the artinian property from R. By Example 1.3, this forces A to be a field.

Next, we show that there are only finitely many prime ideals in R. All primes are maximal, so for any finite collection of distinct prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ we have $J := \cap \mathfrak{p}_i = \prod \mathfrak{p}_i$ by the Chinese Remainder Theorem for pairwise comaximal ideals. If $\mathfrak{p}_{n+1}$ is an additional prime ideal not in our list then the intersection $J \cap \mathfrak{p}_{n+1}$ is *strictly smaller* than J. Indeed, if not then

$$\prod_{1 \leq i \leq n} \mathfrak{p}_i = J \subset \mathfrak{p}_{n+1},$$

but by choosing $x_i \in \mathfrak{p}_i$ not contained in $\mathfrak{p}_{n+1}$ for $1 \leq i \neq n$ (as we may do since $\mathfrak{p}_i \not\subset \mathfrak{p}_{n+1}$) we then have $\prod x_i \in \prod_{1\leq i\leq n} \mathfrak{p}_i \subset \mathfrak{p}_{n+1}$, contradicting the primality of $\mathfrak{p}_{n+1}$ (since $x_i \notin \mathfrak{p}_{n+1}$ for all $1 \leq i \leq n$).

It follows that if there exist infinitely many distinct prime ideals then by enumerating a countably infinite sequence of such and forming the intersections $J_n = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$ we get a s decreasing chain of ideals that does not stabilize. This contradicts dcc for R. ■

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the finite set of prime ideals of the artinian ring R. Since they are each maximal, $J := \cap \mathfrak{p}_i$ is equal to $\prod \mathfrak{p}_i$. In any commutative ring the intersection of all prime ideals is the nilradical (as we saw on HW5), so J is the nilradical of R.

Lemma 2.2. *For some large e, the ideal J^e vanishes.*

Proof. We first use localization to reduce to the case when there is only one prime ideal. It suffices to check the vanishing of J^e for large e after localizing at each of the *finitely many* maximal ideals $\mathfrak{p}_i$ of R (why?), and $J_{\mathfrak{p}_i}$ is the maximal ideal $\mathfrak{p}_i R_{\mathfrak{p}_i}$ of $R_{\mathfrak{p}_i}$ (why?). Thus, provided that the localization $R_{\mathfrak{p}_i}$ is artinian, we may assume there is only one prime ideal.

Rather generally, if S is any multiplicative set of R then we claim that $S^{-1}R$ is artinian. Any ideal I in $S^{-1}R$ is generated by the S-localization of its preimage under $R \to S^{-1}R$ (roughly speaking, chase numerators of fractions), so any descending chain in $S^{-1}R$ comes from one in R. Thus, $S^{-1}R$ inherits dcc for ideals from R, as desired.

Now we may and do assume R has a unique prime ideal $\mathfrak{p}$. In particular, $R - \mathfrak{p} = R^\times$. We want to show that $\mathfrak{p}^e = 0$ for large e. Since the powers of $\mathfrak{p}$ constitute a descending chain of ideals in R, by dcc certainly $\mathfrak{p}^e = \mathfrak{p}^{e+1}$ for some large e. We claim that for this exponent e, necessarily $\mathfrak{p}^e = 0$. Assuming to the contrary, we will get a contradiction.

Consider the collection Σ of ideals $I \subset R$ such that $I\mathfrak{p}^e \neq 0$; e.g., $I = (1)$ is such an ideal under our new hypothesis about $\mathfrak{p}^e$, so Σ is non-empty. By the dcc condition, Σ must contain an ideal I that is minimal with respect to inclusion among members of Σ (as otherwise we could violate dcc for R). Clearly

$$(I\mathfrak{p})\mathfrak{p}^e = I\mathfrak{p}^{e+1} = I\mathfrak{p}^e \neq (0),$$

so $I\mathfrak{p} \in \Sigma$. But $I\mathfrak{p} \subset I$, so minimality forces $I = I\mathfrak{p}$.

There exists $a \in I$ such that $(a)\mathfrak{p}^e \neq (0)$ since $I\mathfrak{p}^e \neq (0)$, so $(a) \in \Sigma$ and hence $(a) = I$ by minimality of I in Σ. Thus, $(a) = (a)\mathfrak{p}$. This implies $a = ax$ for some $x \in \mathfrak{p}$, so $a(1-x) = 0$. But $1 - x \in R - \mathfrak{p} = R^\times$, so $a = 0$ and hence $I = 0$. This contradicts that $I\mathfrak{p}^e \neq (0)$. ■

Proposition 2.3. *The artinian ring R is noetherian.*

Proof. Let $J = \prod \mathfrak{p}_i = \cap \mathfrak{p}_i$ as above, so $J^e = 0$. Hence, we can make a finite sequence (allowing repetitions!) of prime ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_N$ so that $\prod \mathfrak{q}_j = 0$. Since chain of ideals $\{I_m\}_{m\geq 0}$ for $I_m = \prod_{j=1}^m \mathfrak{q}_j$ terminates at 0 for large m, to show R is a noetherian R-module it suffices to show that each successive quotient I_m/I_{m+1} is a noetherian R-module. This quotient is an artinian module (since R is artinian as an R-module), and it is also a vector space over the field $R/\mathfrak{q}_{m+1}$. It subspaces over this field are *the same* as its R-submodules, and an artinian vector space is always noetherian (Example 1.2), so we are done! ■

To summarize, we have shown that an artinian commutative ring is noetherian and that its prime ideals are both maximal and minimal. In HW6 Exercise 5(ii) you will show the converse, so this provides a characterization of artinian rings among all noetherian rings. (In terminology to come later, artinian rings are precisely the zero-dimensional noetherian rings.)