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Introduction. We shall first review the topic of algebraic geometry from its naive beginnings to the curious and outlandish fashion in which it is treated nowadays.

In the elements of calculus a plane curve is defined as the set of points $(x,y)$ which satisfy the equation $P(x,y) = 0$, where P is a polynomial with coefficients in the real field $\mathbb{R}$. Similarly a surface in 3-space is the set of points $(x,y,z)$ which satisfy the equation $P(x,y,z) = 0$, where $P \in \mathbb{R}[X,Y,Z]$. There exist, however, curves which according to this definition have no points at all, like for instance $x^2 + y^2 + 1 = 0$. The mathematicians of the 18th century thought to eliminate this incbonvenience by introducing imaginary points of the curve, i.e. points $(x,y) \in \mathbb{C}^2$ which satisfy $P(x,y) = 0$. They created hereby a great confusion since considering complex points of curves defined by equations with real coefficients they mixed up two completely different problems. We now know that the right thing to do is to consider a field K, an equation $P(x,y) = 0$ where P is a polynomial with coefficients in K and to look for solutions $(x,y) \in K^2$. The curves defined over different fields are to be considered as different curves, even if their equations have the same form.

What is the reason for considering an arbitrary field K instead of just the fields $\mathbb{R}$ and $\mathbb{C}$ ? A great number of problems have conduced to arbitrary fields, especially since the year 1930. Thus Fermat's last theorem states that the surface $x^n + y^n + z^n = 0$ $(n \geqslant 3)$ has no points over the field $\mathbb{Q}$ of rational numbers. The situation is similar for all diophantine equations. An important method in the theory of these equations is the reduction modulo p, where p is a prime number. Thus in the case of Fermat's last theorem we ask whether the congruence

$$x^n + y^n + z^n \equiv 0 \pmod p$$

has a solution in integers, which is the same thing as to ask for solutions of the equation $x^n + y^n + z^n = 0$ over the finite field $\mathbb{F}_p = \mathbb{Z}/(p)$. We can also consider congruences modulo $p^k$, i.e. solutions in $\mathbb{Z}/(p^k)$ which is not a field any more but a ring. Finally we can ask for solutions which satisfy the congruences mod $p^k$ for every strictly positive integer k, which is the same as asking for solutions of $x^n + y^n + z^n = 0$ in the ring of p-adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/(p^k)$ = the projective limit of the rings $\mathbb{Z}/(p^k)$. Hence we are led to consider the field $\mathbb{Q}_p$ of p-adic numbers.

The method of algebraic geometry over finite fields with characteristic p is very powerful. Thus after André Weil proved the Riemann conjecture for algebraic curves, he has obtained a number of immediate corollaries concerning diophantine equations which have never been proved before. For instance he obtained the inequality

$$\left| \sum_{x=1}^{p-1} e^{\frac{2\pi i}{p}\left(x + \frac{1}{x}\right)} \right| \leqslant c \sqrt{p}.$$

The best estimate known before was $O(p^{\alpha})$ with $1/2 < \alpha < 1$.

Thus we are led to consider an arbitrary field K and points of $K^n$ which satisfy a certain family of polynomial equations $P_{\alpha}(x_1,\ldots,x_n) = 0$ $(\alpha \in I)$ with coefficients in K. The set V of all such points form a so-called geometric or affine variety. The first remark to be made is that while the polynomials $P_{\alpha}$ determine the variety V, conversely the variety in no way determines the system $P_{\alpha}$ since the points of V satisfy all equations of the form $\Sigma\, R_{\alpha}P_{\alpha} = 0$. This was noticed at a very early stage since already in three-dimensional space a curve can be defined be infinitely many different systems of equations. Thus we are led to consider, instead of systems of polynomial equations, ideals in the polynomial ring. Given an ideal $\mathfrak{a}$ in $K[X_1,\ldots,X_n]$ let $V(\mathfrak{a})$ be the set of all points $(x_1,\ldots,x_n) \in K^n$ which satisfy the equation $P(x_1,\ldots,x_n) = 0$ for every $P \in \mathfrak{a}$. Conversely let A be any set in $K^n$ and consider the set I(A) of all polynomials P such that $P(x_1,\ldots,x_n) = 0$ for all $(x_1,\ldots,x_n) \in A$. The set $I(A)$ is clearly an ideal and thus we have associated ideals and varieties to each other. This association is, however, not very satisfactory. Indeed, if we start out with an ideal $\mathfrak{a}$, then $\mathfrak{a} \subset I(V(\mathfrak{a}))$, but in general $\mathfrak{a} \neq I(V(\mathfrak{a}))$.

The situation is slightly better if the field K is algebraically closed, for in this case we have Hilbert's nullstellensatz: if the ideal $\mathfrak{a}$ is not the whole ring $K[X_1,\ldots,X_n]$, then there exists a point $x = (x_1,\ldots,x_n) \in K^n$ such that $x \in V(\mathfrak{a})$, i.e. $V(\mathfrak{a}) \neq \emptyset$. From here we can deduce using the classical trick of Rabinowitsch that

$$I(V(\mathfrak{a})) = \mathfrak{r}(\mathfrak{a}),$$

where $\mathfrak{r}(\mathfrak{a})$ is the root of $\mathfrak{a}$, i.e. the set of all polynomials such that some power belongs to $\mathfrak{a}$. Proof. Clearly $\mathfrak{r}(\mathfrak{a}) \subset I(V(\mathfrak{a}))$. Conversely, let $f \in I(V(\mathfrak{a}))$ and let T be a new indeterminate. Let us consider the ideal $\mathfrak{B}$ in the ring $K[X_1,\ldots,X_n,T]$ generated by $1 - T f(X_1,\ldots,X_n)$ and all the polynomials $Q(X_1,\ldots,X_n) \in \mathfrak{a}$. These polynomials have no common root, since if $x = (x_1,\ldots,x_n)$ is a root of all polynomials $Q \in \mathfrak{a}$, then by our assumption $f(x_1,\ldots,x_n) = 0$ and $1 - T f(x_1,\ldots,x_n) = 1$. Hence by the nullstellensatz the ideal $\mathfrak{B}$ is the whole ring $K[X_1,\ldots,X_n,T]$ and thus 1 is a linear combination of the form

$$1 = g(T,X_1,\ldots,X_n)(1 - T f(X_1,\ldots,X_n)) + \sum_i g_i(T,X_1,\ldots,X_n)Q_i(X_1,\ldots,X_n).$$

Setting $T = \frac{1}{f(X_1,\ldots,X_n)}$ we obtain an identity in the field $K(X_1,\ldots,X_n)$:

$$1 = \sum_i g_i\left(\frac{1}{f(X_1,\ldots,X_n)}, X_1,\ldots,X_n\right) Q_i(X_1,\ldots,X_n) .$$

Multiplying by the highest power of $f(X_1,\ldots,X_n)$ which occurs in the denominators on the right hand side we obtain an identity

$$f^{\mu} = \sum_i h_i(X)\, Q_i(X)$$

which proves that $f^{\mu} \in \mathfrak{a}$, i.e. $f \in \mathfrak{r}(\mathfrak{a})$.

It can be seen in particular that $\mathfrak{a}$ is a prime ideal if and only if $V(\mathfrak{a})$ is irreducible. Since in this case $\mathfrak{a} = \mathfrak{r}(\mathfrak{a})$, the correspondence between $\mathfrak{a}$ and and V is bijective. Thus in the hands of Hilbert and his followers (E. Noether, W. Krull, B. L. van der Waerden) algebraic geometry has become the study of polynomial ideals. This school has flourished from 1920 to the publication of Weil's Foundations in 1947 and its results can be found in the books of van der Waerden (Einführung in die algebraische Geometrie, 1939) and Gröbner (Moderne algebraische Geometrie, 1949).

However, the situation is still not satisfactory. The main objection is that what the theory really studies is not the algebraic variety itself but the algebraic variety immersed in a certain space. Thus a circle if considered in the plane is defined by the ideal $\mathfrak{a}$ generated by $X^2 + Y^2 - 1$ in $K[X,Y]$. However, the same circle, if considered in 3-space, is defined by the ideal generated by the polynomials $X^2 + Y^2 - 1$ and Z in $K[X,Y,Z]$. The solution to this dilemma has been seen already by Riemann and it consists in considering the quotient ring $A = K[X_1,\dots,X_n]/\mathfrak{a}$; in the above two examples the two rings obtained in this way are isomorphic. The elements of the ring A are polynomials computed mod $\mathfrak{a}$, i.e. two polynomials $P_1$ and $P_2$ define the same elements of A if they define the same function on V. Thus the ring A can be thought of as the ring of functions on the variety V.

Riemann expressed himself, of course, in a different language. He considered a rational function $f(x,y)$ and for every fixed x the roots $y_1(x),\dots,$ $y_n(x)$ of the equation $f(x,y) = 0$. The value of a rational function at a point $(x,y_1)$ is then given by $\frac{P(x,y_1(x))}{Q(x,y_1(x))}$. The trouble with this language is that the $y_i(x)$ are multiple valued functions, i.e. no functions at all. One really has to work with the integral ring $K[X_1,\dots,X_n]/\mathfrak{p}$, $\mathfrak{p}$ prime, and its quotient field. Later Dedekind and Weber translated Riemann's theory into algebraic language (Journ. für Reine Angew. Math., 92 (1882) pp. 181-290).

Thus we are led to associate with every variety V a ring of finite type $A = K[X_1,\dots,X_n]/\mathfrak{a}$, where, for the sake of simplicity, we assume that the field K is algebraically closed. Our problem is to establish a one-to-one correspondence between algebraic and geometric objects. Given a point $z \in V$, we associate with it the set of all functions on V which vanish at z; this set is a maximal ideal in A. A subvariety W of V is defined by an ideal $\mathfrak{b}$ which contains $\mathfrak{a}$. The set of all functions on V which vanish on W forms an ideal in A. We associate this ideal with the subvariety W.

In the opposite direction, given an algebraically closed field K and a ring of finite type $A = K[X_1,\dots,X_n]/\mathfrak{a}$, we want to associate with it a variety V. By what precedes, it is natural to consider as the points of V the elements of the maximal spectrum Specm(A) of A, i.e. the maximal ideals of A. This, however, will certainly not give a one-to-one correspondence between algebraic and geometric objects, since to any field K there would correspond the variety consisting of one point. A way to correct this situation is again suggested by Riemann's approach who considered the rings $A_z$ formed by the functions on V which have no poles at z. Thus we should take as elements of the geometric object we want to associate with the ring A, the pairs $(\mathfrak{m}, A_{\mathfrak{m}})$, where $\mathfrak{m}$ is a maximal ideal of A and $A_{\mathfrak{m}}$ is the local ring at $\mathfrak{m}$. In this way for different fields we obtain different pairs (z,K).

The situation is still not satisfactory, since there exist non-isomorphic rings such that the corresponding sets of pairs $(\mathfrak{m}, A_{\mathfrak{m}})$ are the same. For example if we take for varieties over $\mathbb{C}$ the hyperbola $V_1$ defined by the equation $xy = 1$ and the parabola $V_2$ defined by $y = x^2$, then the corresponding rings are $A_1 = \mathbb{C}[X,Y]/(XY - 1)$ and $A_2 = \mathbb{C}[X,Y]/(X^2 - Y)$. Now $A_1$ is isomorphic to $\mathbb{C}[X,X^{-1}]$, $A_2$ is isomorphic to $\mathbb{C}[X]$ and the last two rings are not isomorphic since in the second one the sum of invertible elements is an invertible element or zero and in the first one this is not true. However, it is easy to see that for either ring the local ring at a maximal ideal is isomorphic to $(\mathbb{C}[X])_{(X)}$ = the local ring of $\mathbb{C}[X]$ at the maximal ideal (X).

The way out of this dilemma is to glue together the pairs $(\mathfrak{m}, A_{\mathfrak{m}})$ in a sensible way instead of considering them as a discrete collection of objects. This can be done using Leray's theory of sheaves. First we define a topology on the set Specm(A), the so-called Zariski topology. (Observe that in the case of the real or complex field, we have on the variety the topology induced by $\mathbb{R}^n$ or $\mathbb{C}^n$, but this cannot be generalized to an arbitrary field K.) Given an ideal $\mathfrak{a}$ in A we set $V(\mathfrak{a})$ = the set of all maximal ideals which contain $\mathfrak{a}$ and we take for the closed sets of the Zariski topology the sets $V(\mathfrak{a})$. In other words, the closed sets of V = Specm(A) are the subvarieties of V. The Zariski topology is very coarse, it is $T_1$ (Fréchet) but never $T_2$ (Hausdorff). Now we want to construct on Specm(A) a sheaf whose stalk at the point $\mathfrak{m}$ is $A_{\mathfrak{m}}$. We take for a basis of the open sets in V the sets of the form $D(f) = \{\mathfrak{m} \mid f \notin \mathfrak{m}\}$, where $f \in A$. The sets D(f) can also be interpreted in another way. Write now x for an arbitrary element of V = Specm(A) and denote by f(x) the class of $f \in A$ modulo the maximal ideal $\mathfrak{m}_x$ corresponding to x. In particular $f(x) = 0$ if and only if $f \in \mathfrak{m}_x$. (This corresponds to the fact that if A is the ring of functions on the variety V, then with each point $x \in V$ we associate the maximal ideal of A formed by the functions which vanish at x.) Then we can also write $D(f) = \{x \mid f(x) \neq 0\}$. We obtain a presheaf of rings $\widetilde{A}$ on Specm(A) if we attach to every open set D(f) the ring $\Gamma(D(f), \widetilde{A}) = A_f = \left\{ \frac{x}{f^n} \mid x \in A, n \geq 0 \right\}$ (Grothendieck, Eléments, $0_I$, 1.2.3). Intuitively $A_f$ is the set of functions which have poles at most on the set $V(f) = \{x \mid f(x) = 0\}$ where f vanishes, i.e. which are regular on D(f). The restriction maps for $\widetilde{A}$ are readily defined and the axioms of sheaves are easily checked. Thus we have attached to the ring A a geometric object, namely the ringed space $(V, \widetilde{A})$. It is easy to see that $A_{\mathfrak{m}}$

is the inductive limit of the rings $A_f$, where $f \notin \mathfrak{m}$. Finally A is isomorphic to $\Gamma(V,\tilde{A})$ and thus the correspondence between rings A and ringed spaces $(V,\tilde{A})$ is one-to-one.

We have now obtained a satisfactory solution to our problem in the case of affine geometry. However, already the geometers of the $18^{th}$ century have realized that affine geometry does not reflect correctly the intuitive geometric picture. Thus similar objects, like conic sections, look completely different and Bezout's theorem, according to which a curve of degree m and a curve of degree n have mn points of intersection if counted with their multiplicities, does not hold (e.g. a line and a conic). Early in the $19^{th}$ century projective space was introduced and there general theorems (e.g. Bezout's theorem, at least over an algebraically closed field) were found to be true. Consequently in the ensuing period mostly projective algebraic geometry was developed. Consider the n-dimensional projective space $\mathbb{P}_n(K)$ over the field K which is defined as the quotient set $K^{n+1}/\Delta$ where $\Delta$ is the equivalence relation in $K^{n+1}$ whose equivalence classes are lines going through the origin. The varieties in $\mathbb{P}_n(K)$ correspond to cones in $K^{n+1}$, i.e. are defined by a system of equations $P_\alpha(x_o,x_1,\ldots,x_n) = 0$ $(\alpha \in I)$, where the polynomials $P_\alpha$ are homogeneous. The first trouble arises if, following the pattern set by the affine case, we want to define the functions on the variety V as equivalence classes of polynomials, since all such functions turn out to be constants. Once more it was Riemann who circumvented this difficulty. Instead of considering functions which are regular on the whole variety, he allowed poles, i.e. considered rational functions. Two such rational functions

$$\frac{Q(x_o,x_1,\ldots,x_n)}{R(x_o,x_1,\ldots,x_n)} \quad \text{and} \quad \frac{Q'(x_o,x_1,\ldots,x_n)}{R'(x_o,x_1,\ldots,x_n)}$$

were considered to be equal if $QR' - RQ'$ belonged to the homogeneous ideal generated by the polynomials $P_\alpha$. It turns out that these rational functions form a field with a reasonable degree of transcendence. Something is lost however: isomorphic fields can correspond to different varieties, as in the case of rational (unicursal) curves.

This method of attack has given rise to birational geometry, which disregards the finer geometric properties of the objects under study but has given a great number of interesting results in the hands of German, French and Italian geometers. Birational invariants were discovered, the first of them by Riemann himself, namely the genus of a curve.

To go beyond this stage it is necessary to consider projective space in a different way. In fact $\mathcal{P}_n(K)$ can be obtained by gluing together affine spaces. Let $H_i$ be the hyperplane defined by $x_i = 0$ $(0 \leqslant i \leqslant n)$, then $E_i = \complement H_i$ is isomorphic to $K^n$, the isomorphism being defined by $(x_o, x_1, \ldots, x_i, \ldots, x_n) \to \left( \frac{x_o}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right)$. The set $E_i \cap E_j$ is an open variety in both $E_i$ and $E_j$. Thus to obtain $\mathcal{P}_n(K)$ we have to take $n + 1$ affine spaces $E_i$, in each of these certain subvarieties $H_{ij}$ $(j \neq i)$ and identify $H_{ij} \subset E_i$ with $H_{ji} \subset E_j$. This procedure has been used before in algebraic topology and to define differentiable or analytic manifolds, but the idea to use it in algebraic geometry is due to André Weil. Once in possession of this idea Weil introduced abelian varieties (which turned out much later to be projective varieties) and could prove the Riemann hypothesis for algebraic curves over finite fields. To define his abstract varieties Weil considers a system $V_\alpha$ of algebraic varieties and for each pair $V_\alpha$, $V_\beta$ subvarieties $W_{\alpha\beta} \subset V_\alpha$ and $W_{\beta\alpha} \subset V_\beta$, such that $W_{\alpha\beta}$ and $W_{\beta\alpha}$ are isomorphic. The abstract variety is then obtained by identifying $W_{\alpha\beta}$ and $W_{\beta\alpha}$. Weil could extend to these varieties all the results known for affine and projective varieties. However, the detailed construction of the abstract varieties is an extremely long, tedious, and cumbersome process.

Just as in the case of differentiable and analytic manifolds, the simplification was achieved by the use of sheaves. A differentiable manifold can be defined as a ringed space $(V, \mathcal{F})$ in which every point $x \in V$ has a neighborhood U such that $\mathcal{F}|U$ is isomorphic to the sheaf of all differentiable functions on an open set of $\mathbb{R}^n$. This definition has the advantage to be intrinsic and can easily be seen to be equivalent to the old one with atlases. In 1955 J.P. Serre has turned his attention from complex analytic manifolds and analytic spaces to algebraic geometry and found that the same definition can be given here (Faisceaux algébriques cohérents, Annals of Math. 61 (1955) pp. 197-278). Thus an algebraic variety is a ringed space which is locally isomorphic to the ringed space of an affine variety. This definition was finally modified by Grothendieck in two points.

In the first place Serre still only considered rings of finite type $A = K[X_1,\ldots,X_n]/\mathfrak{a}$ (Weil even took $\mathfrak{a}$ prime, i.e. his varieties were irreducible). Grothendieck (and independently Cartier) had the idea to consider a completely arbitrary commutative ring A with unit element.

Another point still caused some difficulty. With a ring A we associated a ringed space $(V,\tilde{A})$ and it is desirable that this correspondence be a functor from the category of rings into the category of ringed spaces. A morphism $\Psi : (V, \mathcal{O}) \to (V', \mathcal{O}')$ of ringed spaces is a pair $\Psi = (\psi, \theta)$, where $\psi : V \to V'$ is a continuous map and $\theta : \mathcal{O}' \to \mathcal{O}$ is a $\psi$-morphism (Grothendieck, Eléments, $0_I$,4.1.1). If $\varphi : A \to B$ is a homomorphism of rings (transforming, as always, the unit element of A into the unit element of B), then the only sensible way to define $\psi : W = \mathrm{Specm}(B) \to V = \mathrm{Specm}(A)$ is as follows: take $\mathfrak{n} \in \mathrm{Specm}(B)$, i.e. let $\mathfrak{n}$ be a maximal ideal in B and consider $\mathfrak{m} = \varphi^{-1}(\mathfrak{n})$. It turns out, however, that $\varphi^{-1}(\mathfrak{n})$ is not necessarily a maximal ideal of A at all. Thus if $i : \mathbb{Z} \to \mathbb{Q}$ is the canonical imbedding, (0) is a maximal ideal in $\mathbb{Q}$, but $i^{-1}(0) = (0)$ is not a maximal ideal in $\mathbb{Z}$.

In this way the maximal spectrum Specm(A) had to be replaced by the prime spectrum Spec(A) and now $\varphi : A \to B$ defines a map $\psi :$ Spec(B) $\to$ Spec(A) since if $\mathfrak{p}$ is a prime ideal in B, then $\varphi^{-1}(\mathfrak{p})$ is a prime ideal in A. The intuitive picture is somewhat more confusing since maximal ideals correspond to points, whereas we include also all irreducible subvarieties, which correspond to prime ideals. It is clear that for each $\mathfrak{p} \in$ Spec(A) we have a homomorphism $\varphi_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ which together define a $\psi$-morphism $\widetilde{A} \to \widetilde{B}$ of sheaves of rings. Thus $A \rightsquigarrow$ (Spec(A), $\widetilde{A}$) is indeed a functor as desired (Grothendieck, Eléments, I,1.6.1).

Interpretation of some notions of classical algebraic geometry in the language of schemata. As we have mentioned before, the points of a preschema $(X, \mathcal{O}_X)$ are not only the points of the variety V corresponding to it but also all irreducible subvarieties of V. The points of X which correspond to points of V are the closed points x of X, i.e. such that $\{x\} = \overline{\{x\}}$ . In the case of an affine schema Spec(A) the closed points x correspond to maximal ideals $\mathfrak{j}_x$ of A (Grothendieck, Eléments, I,1.1.7). The variety in the sense of Serre is the subspace formed by all closed points with the topology induced on it by X.

To an irreducible subvariety of the affine variety V there corresponds a prime ideal $\mathfrak{a}$ of A and thus an irreducible closed set $V(\mathfrak{a})$ of X = Spec(A) (Grothendieck, Eléments, I,1.1.14). Each irreducible closed set F in X has a unique generic point x such that $F = \overline{\{x\}}$ , namely the point $x = \mathfrak{j}_x = \mathfrak{j}(F)$ (ibid.). All points $y \in \overline{\{x\}}$ are specializations of x, i.e. $\mathfrak{j}_x \subset \mathfrak{j}_y$ or, in other words, the irreducible subvariety $W_y$ corresponding to $\mathfrak{j}_y$ is a subvariety of $W_x$. This notion of generic point is completely different from that of André Weil.

Next we consider the notion of dimension. Let E be a topological space. We consider increasing chains $F_0 \subset F_1 \subset \ldots \subset F_k$ of length k of irreducible, non-empty, closed sets in E. The dimension of E is by definition the maximal length of all such chains (Godement, Théorie des faisceaux, II.4.15, p.198). This definition corresponds to the intuitive idea of dimension in an algebraic variety. Thus for instance in a three dimensional variety we have maximal chains consisting of a point, a curve through the point, a surface through the curve and finally the space itself. If the space is an affine schema, then each irreducible closed set $F_i$ has a unique generic point $x_i$, hence the chain can be written $\overline{\{x_o\}} \subset \overline{\{x_1\}} \subset \ldots \subset \overline{\{x_k\}}$ and to this there corresponds a chain of prime ideals $\mathfrak{p}_o \supset \mathfrak{p}_1 \supset \ldots \supset \mathfrak{p}_k$ in A. Irreducible components of the variety correspond to minimal prime ideals of A (cf. Grothendieck, Eléments, I,1.1.14). Given an irreducible variety W corresponding to a prime ideal $\mathfrak{p}$ of A, the dimension of W is the length of the maximal chain $\mathfrak{p} = \mathfrak{p}_k \subset \mathfrak{p}_{k-1} \subset \ldots \subset \mathfrak{p}_o$ of strictly increasing prime ideals of A.

This leads to the definition of the (Krull) dimension of a ring A as the largest number k such that there exists a strictly increasing chain of prime ideals $\mathfrak{p}_o \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_k$ of A (Samuel, Progrès récents d'algèbre locale, III.2, p.66). We can now speak of the dimension of the local ring $\mathcal{O}_x = A_{\mathfrak{p}}$ at $\mathfrak{p}$. At a generic point x the dimension is zero, since $A_{\mathfrak{p}}$ has only one prime ideal, namely the nilradical. If X = Spec(A) is irreducible and A is an integral ring, than at a generic point x the local ring $\mathcal{O}_x$ is a field, namely the field of fractions of A (Grothendieck, Eléments, I,7.1.5). Corresponding to a maximal chain $\mathfrak{p}_o \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_k$ of prime ideals of A we have a sequence of local rings $A_{\mathfrak{p}_o}, A_{\mathfrak{p}_1}, \ldots, A_{\mathfrak{p}_k}$, where $\dim A_{\mathfrak{p}_o} = 0$, $\dim A_{\mathfrak{p}_1} = 1, \ldots, \dim A_{\mathfrak{p}_k} = k$.

Analyzing the classical definition of a simple point of a variety (Zariski, The concept of a simple point of an abstract algebraic variety, Trans. Amer. Math. Soc. 62 (1947) pp. 1-52, or Lang, Introduction to algebraic geometry, Chap. VIII) one is led to say that $x \in X$ is simple if $\mathcal{O}_x$ is a regular local ring. Let us recall the definition of these rings. Let A be a noetherian local ring and $\mathfrak{m}$ its maximal ideal. Let n be the minimal number of generators of $\mathfrak{m}$ and d the dimension of A. In general $d \leqslant n$, if $d = n$ we say that A is a regular local ring. Observe that n is also the dimension of the vector space $\mathfrak{m}/\mathfrak{m}^2$ over the field $A/\mathfrak{m}$. This follows from Nakayama's lemma, which states the following: let A be a not necessarily commutative ring, $\mathcal{R}$ the radical of A, M a module of finite type over A. If N is a submodule of M such that $M = N + \mathcal{R} M$, then $M = N$ (Bourbaki, Alg., Chap. VIII, § 6, n° 3, cor. 2 of prop. 6). To prove the assertion, let $x_1, \ldots, x_n \in \mathfrak{m}$ be such that their classes $\bar{x}_1, \ldots, \bar{x}_n$ modulo $\mathfrak{m}^2$ generate $\mathfrak{m}/\mathfrak{m}^2$, then we have to show that $x_1, \ldots, x_n$ generates $\mathfrak{m}$. Set $\mathfrak{n} = Ax_1 + \ldots + Ax_n$, then $\mathfrak{n} \subset \mathfrak{m}$ and $\mathfrak{m} = \mathfrak{n} + \mathfrak{m}^2$. Since $\mathfrak{m}$ is the radical of A, we obtain by Nakayama's lemma that $\mathfrak{m} = \mathfrak{n}$.

Intuitively, if A is the ring of functions regular at a point x of a variety and $\mathfrak{m}$ is its maximal ideal consisting of the functions which vanish at x, then $\mathfrak{m}^2$ is the ideal consisting of those functions whose development starts with quadratic terms and the elements of $\mathfrak{m}/\mathfrak{m}^2$ are the equivalence classes of functions with the same linear terms and no constant terms. The fact that A is a regular local ring means that the dimension of the vector space spanned by the gradients at x of the functions vanishing at x equals the dimension of the variety.

To see an example of a non-simple point, consider the curve $x = t^2$, $y = t^3$, which has a cusp at the origin (0,0). The polynomial ring $K[X,Y]$ has dimension two, however, this fact is not so trivial as one might think (Samuel, Progrès récents d'algèbre locale, III.3, Th. 8, p. 83). Indeed, Nagata has given an example of a non-noetherian ring R of dimension 1 such that $R[X]$ has dimension three. The ring $A = K[X,Y]/(Y^2 - X^3)$ has dimension 1, and the maximal ideal $(X) + (Y)/(Y^2 - X^3)$ of the functions vanishing at the origin can be generated by two elements but not by one.