Math 210B. Canonicity of local decomposition of artinian rings

1. Motivation

In HW6 Exercise 5 it was shown that (nonzero) artinian (commutative) rings are precisely the 0-dimensional noetherian rings, and that for any such ring A there is a *canonical* direct product decomposition

$$A \simeq \prod A_{\mathfrak{m}}$$

where $\mathfrak{m}$ varies through the finitely many maximal ideals of A (which are all the prime ideals of A) and each $A_{\mathfrak{m}}$ is a *local* artinian ring (with maximal ideal that is nilpotent). Conversely, any such product was seen to be a 0-dimensional noetherian ring, hence artinian.

As a special case, if A is reduced (i.e., no nonzero nilpotents) then each $A_{\mathfrak{m}}$ must be reduced and so its nilpotent maximal ideal vanishes. That is, each $A_{\mathfrak{m}}$ is a *field*, so in the reduced case

$$A \simeq \prod_{i=1}^{n} k_i$$

for fields k_i. Conversely, any finite product of fields is visibly a reduced artinian ring.

The purpose of this handout is to explain the sense in which this direct product decomposition is both "unique up to rearrangement" as well as functorial with respect to *isomorphisms* in A. For example, we will see that if $\{k_1, \dots, k_n\}$ is a finite (non-empty) collection of fields then the *ring* $A = \prod k_i$ intrinsically determines the collection of k_i's up to rearrangement as exactly the residue fields at its maximal ideals (in contrast with the fact that if k is a single field then the k-algebra k^n viewed just as a k-vector space in no way recovers the evident factor fields just from the linear algebra structure; the ring structure is essential).

2. Decomposing isomorphisms into local parts

Suppose we are given two non-empty finite collections of artin *local* rings $\{A_1, \dots, A_n\}$ and $\{B_1, \dots, B_m\}$ and a ring isomorphism

$$\varphi : A := \prod A_i \simeq \prod B_j =: B.$$

We claim that necessarily $m = n$ and that, after unique relabeling, there are uniquely determined ring isomorphisms $\varphi_i : A_i \simeq B_i$ for all i such that $\varphi = \prod \varphi_i$. (The situation for general ring homomorphisms between such products is more delicate.)

Note that the topological space $X = \mathrm{Spec}(A)$ is a non-empty finite discrete space. We claim that its points correspond to the local factor rings A_i (or more specifically, to the kernels of the projections of A onto the residue fields k_i of A_i for all i). Naturally $\mathrm{Spec}(A) = \mathrm{Spec}(A_{\mathrm{red}})$ where A_{red} is the quotient by the ideal of nilpotent elements, and $A_{\mathrm{red}} = \prod k_i$ for the residue fields $k_i = A_i/\mathfrak{m}_i$, so we just have to check:

Lemma 2.1. *For fields $k_1, \dots, k_n$, the maximal ideals of the artinian ring $P = \prod k_i$ are exactly the kernels of the projections $P \twoheadrightarrow k_i$.*

Proof. Consider the elements $e_i = (0, \dots, 1, \dots, 0)$ (1 in the ith slot, 0 elsewhere). These are idempotent (i.e., $e_i^2 = e_i$ for all i) and orthogonal (i.e., $e_i e_j = 0$ for all $i \neq j$) and $\sum e_i = 1$. Hence, under any quotient map $P \twoheadrightarrow k$ onto a field each e_i must map to 0 or 1 (the only idempotents in k) yet since they are pairwise orthogonal at most one of them can map to 1. But their sum is 1, so exactly one of them does map to 1. That is, this quotient map carries some e_{i_0} to 1 and kills all other e_i's. The elements of P that arise from the ith factor (in the sense of vanishing along the other factors) are exactly the multiples of e_i in P, so the quotient map onto k must factor through the natural quotient map onto k_{i_0}. Since surjections between fields (such as $k_{i_0} \twoheadrightarrow k$) are isomorphisms, we are done. ■

Consider the map $f : \mathrm{Spec}(B) \simeq \mathrm{Spec}(A)$ induced by φ. This is a homeomorphism between finite discrete spaces, the points of which we have seen correspond to the factor rings. Thus, we can label the points of $X = \mathrm{Spec}(A)$ as $\{x_1, \dots, x_n\}$ and the points of $Y = \mathrm{Spec}(B)$ as $\{y_1, \dots, y_n\}$ so that $f(y_i) = x_i$. Let $\mathfrak{m}_i \subset A$ and $\mathfrak{n}_i \subset B$ be the maximal ideals respectively corresponding to x_i and y_i.

Lemma 2.2. *The localization map $A \to A_{\mathfrak{m}_i}$ is uniquely identified with the quotient map $A \to A_i$.*

Proof. The idempotent e_i maps to 1 in k_i and hence is not in $\mathfrak{m}_i = \ker(A \twoheadrightarrow k_i)$, so it becomes a unit in $A_{\mathfrak{m}_i}$. This forces the orthogonal idempotents e_j for $j \neq i$ to have vanishing image in $A_{\mathfrak{m}_i}$, so the localization map $A \to A_{\mathfrak{m}_i}$ factors through the quotient map $A \to A_i$. This thereby identifies $A_{\mathfrak{m}_i}$ as a localization of A_i at a maximal ideal (due to the compatibility of quotients and localization), yet A_i has only one maximal ideal (since it is assumed to be local)! Hence, the induced map $A_i \to A_{\mathfrak{m}_i}$ of A-algebras is an isomorphism. This is the only such A-algebra isomorphism, in view of unique mapping properties of quotients and localizations. ■

We see similarly that there is a unique B-algebra isomorphism $B_i \simeq B_{\mathfrak{n}_i}$ for all i. Since the map $f = \mathrm{Spec}(\varphi)$ carries y_i to x_i, in terms of functions on Spec we see that composition with φ carries the elements of A which vanish at x_i (i.e., the elements of $\mathfrak{m}_i$) over to elements of B which vanish at y_i (i.e., the elements of $\mathfrak{n}_i$). In other words, the isomorphism $\varphi : A \simeq B$ carries $\mathfrak{m}_i$ over to $\mathfrak{n}_i$ and hence induces an isomorphism $\varphi_i : A_i \simeq B_i$ between the localizations at $\mathfrak{m}_i$ and $\mathfrak{n}_i$ respectively. It remains to show:

Lemma 2.3. *The map $\prod \varphi_i : \prod A_i \simeq \prod B_i$ is φ.*

Proof. To prove that two isomorphisms $\prod A_i \rightrightarrows \prod B_i$ coincide, it is the same to check equality after composing with projection to each factor B_i. But we have seen that projection to the factor ring B_i is identified with the localization map at $\mathfrak{n}_i$, and this resulting composite map must then factor through the localization of $A = \prod A_i$ at the prime ideal of A that is the preimage of $\mathfrak{n}_i$. Under φ this preimage is $\mathfrak{m}_i$ by design (due to how we indexed the x's and y's), and under $\prod \varphi_j$ it is also $\mathfrak{m}_i$ (since this direct product map is built from maps between the factor rings having a common index i, and the maximal ideals are precisely the kernels of the projections to the residue fields of such factor rings). In other words, *both* composite maps $A \rightrightarrows B \to B_i$ factor through the localization/quotient map $A \to A_i$ via maps $A_i \rightrightarrows B_i$. By definition this latter map arising from φ is φ_i, and by construction this latter map arising from $\prod \varphi_j$ is also φ_i. ■