Math 210B. Characters for compact Lie groups

1. Basic definitions and invariant differential forms

Let G be a Lie group: a smooth manifold equipped with a group structure such that the multiplication and inversion maps $m : G \times G \to G$ and $\iota : G \to G$ are C^∞ (the former implies the latter by an argument with the implicit function theorem, but we omit that here). For instance, $\mathrm{GL}_n(\mathbf{R})$ and $\mathrm{GL}_n(\mathbf{C})$ are Lie groups since each is an open subset of the respective vector space $\mathrm{Mat}_n(\mathbf{R})$ and $\mathrm{Mat}_n(\mathbf{C})$ on which multiplication is given by polynomials in matrix entries (and inversion by Cramer's formula involving rational functions whose denominators are powers of the determinant polynomial).

It is a non-obvious fact shown in any course or book on Lie groups that all continuous homomorphisms between Lie groups are automatically C^∞ and all closed subgroups of Lie groups are automatically C^∞ submanifolds. In particular, all of the "classical matrix groups" that are closed subgroups of $\mathrm{GL}_n(\mathbf{R})$ or $\mathrm{GL}_n(\mathbf{C})$ are Lie groups, such as: $\mathrm{SL}_n(\mathbf{R})$, $\mathrm{SL}_n(\mathbf{C})$, $\mathrm{Sp}_{2g}(\mathbf{R})$, $\mathrm{Sp}_{2g}(\mathbf{C})$, orthogonal groups $\mathrm{O}(q)$ of non-degenerate quadratic forms $q : \mathbf{R}^n \to \mathbf{R}$ or $q : \mathbf{C}^n \to \mathbf{C}$, unitary groups $\mathrm{U}(h) \subset \mathrm{GL}_n(\mathbf{C})$ of non-degenerate hermitian forms h on $\mathbf{C}^n$, and the "special" variants $\mathrm{SO}(q) = \mathrm{O}(q)^{\det=1}$ and $\mathrm{SU}(h) = \mathrm{U}(h)^{\det=1}$. Likewise, any continuous representation $G \to \mathrm{GL}_n(\mathbf{R})$ or $G \to \mathrm{GL}_n(\mathbf{C})$ is C^∞. (One can also show that Lie groups admit a unique compatible real-analytic structure, with all continuous maps between Lie groups being automatically real-analytic, and develop the notion of *complex Lie group* by working throughout with complex manifolds, but we won't consider such refinements here.)

For compact G we want to use integration against a suitable measure on G to play the role that averaging does for finite G. To make such measures without a long digression into the general theory of Haar measures, we need to carry out some preliminary considerations with differential forms. A global C^∞ differential j-forms ω on G is called *left-invariant* if it is invariant under pullback by all left-multiplications ℓ_g ($g \in G$). Any such ω is uniquely determined by its value $\omega(e) \in \wedge^j(\mathrm{T}^*_e(G))$ at the identity point e. Indeed, since $\ell_{g^{-1}} : g \mapsto e$, we must have

$$\omega(g) = ((\ell_{g^{-1}})^*\omega)(g) = ((\ell_{g^{-1}})^*)_g(\omega(e))$$

for all $g \in G$ (where for the second equality we write $(f^*)_x : \wedge^j(\mathrm{T}^*_{f(x)}(Y)) \to \wedge^j(\mathrm{T}^*_x(X))$ for the effect of pullback by a C^∞-map $f : X \to Y$, so $(f^*\omega)(x) = (f^*)_x(\omega(f(x)))$ for a global smooth j-form ω on Y). Turning this around, any $\omega_e \in \wedge^j(\mathrm{T}^*_e(G))$ can be extended to a global C^∞ left-invariant j-form $\widetilde{\omega}_e$ that is *defined* by the pointwise formula $\widetilde{\omega}_e(g) := ((\ell_{g^{-1}})^*)_g(\omega_e)$; one has to prove that this pointwise definition really is *smooth* in g, but that can be shown by a local calculation using that multiplication and inversion on G are C^∞. In particular, the space of such left-invariant ω is finite-dimensional, being isomorphic to $\wedge^j(\mathrm{T}^*_e(G))$.

For $j = \dim G$ we have that $\wedge^j(\mathrm{T}^*_e(G))$ is a line, it follows that the nonzero left-invariant top-degree smooth differential forms ω on G are unique up to $\mathbf{R}^\times$-scaling. For any $g \in G$, right multiplication r_g *commutes* with all left multiplications (i.e., $(hx)g = h(xg)$ for all $h, x \in G$), so $(r_g)^*(\omega)$ is also left-invariant and smooth. Hence, $(r_g)^*(\omega) = c(g)\omega$ for some constant $c(g) \in \mathbf{R}^\times$ (computed by comparing at the identity point). Clearly $c(g)$ is unaffected by replacing ω with an $\mathbf{R}^\times$-multiple, and since such multiples exhaust all choices for ω it follows that $c(g)$ depends *only* on g and not on the specific choice of ω. It is easy to check that $g \mapsto c(g)$ is a homomorphism.

The function $|c|$ is called the *modulus character* of G. (In some references c is called the "algebraic" modulus character, due to certain considerations with matrix groups over general fields, lying beyond the level of this course.) Note that $|c| = 1$ precisely when the top-degree left-invariant smooth differential forms are also right-invariant, a condition usually called *bi-invariance*. Local calculations show that c is C^∞ (and in particular continuous), so $|c| = 1$ when G is compact since $|c|(G) \subset \mathbf{R}_{>0}$ is a compact subgroup (and there is none except $\{1\}$). Also, c has constant sign on the identity component G^0, so $c = |c|$ on G^0 (as $c(e) = 1$).

We conclude that if G is compact then ω is right-invariant under G^0. However, the potential sign problems for the effect of right-translation on ω by points of G in other connected components really can occur: if $G = \mathrm{O}(q)$ for $q = \sum x_i^2$ on $\mathbf{R}^n$ then one can show that G has exactly two connected components and $c(g) = (-1)^{n-1}$ on the non-identity component.

Remark 1.1. The function c can be computed by clean explicit formulas for any "matrix group". For example if G is the "$ax+b$ group" consisting of invertible matrices $[a, b] := \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ then $c([a, b]) = 1/a$. To explain the computation of c in general requires some preparations with Lie algebras, so here we limit ourselves to remarking that in all of the classical matrix group examples mentioned above except for orthogonal groups it turns out that $c = 1$! (There is a conceptual reason for this, but it involves advanced knowledge of the structure theory of Lie groups; in can be proved by hand in special cases, of course.)

If ω is any smooth top-degree differential form on a smooth manifold M, it defines a Borel measure $|\omega|$ via integration of the *absolute value* of the coefficient function of ω in local coordinates. The use of the absolute value avoids any need for orientations, but this also makes $|\omega|$ generally not be additive in ω. As an illustration, if ω is a nonzero left-invariant top-degree differential form on G then the measure $|\omega|$ is left-invariant (in that its value on A and $gA = \ell_g(A)$ is the same for any Borel set $A \subset G$). On the other hand, this is generally not right-invariant: the measure of Ag is $|c|(g)$ times the measure of A.

For the remainder of this handout, assume G is compact (but not necessarily connected, since we wish to include finite groups as a special case of our discussion!). Thus, for any left-invariant ω, the measure $|\omega|$ is bi-invarant (as $|c| = 1$) even though ω might not be right-invariant under the action of some connected components away from G^0. However, this bi-invariant measure is not uniquely determined since if we scale ω by some $a > 0$ then the measure scales by a. But there is a canonical choice! Indeed, the integral $\int_G |\omega|$ converges to some positive number since G is compact, so by scaling ω it can be uniquely determined up to a sign by the condition $\int_G |\omega| = 1$. This defines a *canonical* bi-invariant measure on G, called the "volume 1 measure". We denote this measure with the suggestive notation $\mathrm{d}g$ (though it is *not* a differential form, and has no dependence on orientations).

Example 1.2. If $G = S^1$ inside $\mathbf{R}^2$ then $\mathrm{d}g = |\mathrm{d}\theta|/2\pi$. If G is a finite group then $\mathrm{d}g$ is the measure that assigns each element of the group the mass $1/|G|$, so $\int_G f(g)\,\mathrm{d}g = (1/|G|)\sum_{g \in G} f(g)$. Thus, integration against $\mathrm{d}g$ in the finite case is precisely the averaging process that pervades the representation theory of finite groups (in characteristic 0).

For an irreducible finite-dimensional $\mathbf{C}$-linear representation $\rho : G \to \mathrm{GL}(V)$, we define the *character* χ_V (or χ_ρ) to be the function $g \mapsto \mathrm{Tr}(\rho(g))$. This is a smooth $\mathbf{C}$-valued function on G since ρ is C^∞ (so its matrix entries relative to a $\mathbf{C}$-basis of V are smooth $\mathbf{C}$-valued functions). Obviously $\chi_{V\oplus W} = \chi_V + \chi_W$, and some other related identities are proved exactly as for finite groups:

$$\chi_{V^*} = \overline{\chi}_V, \quad \chi_{V\otimes W} = \chi_V \cdot \chi_W, \quad \chi_{\mathrm{Hom}(V,W)} = \overline{\chi}_V \cdot \chi_W$$

where $\mathrm{Hom}(V, W)$ denotes the space of $\mathbf{C}$-linear maps $V \to W$ and is equipped with a *left* G-action via $g.T = \rho_W(g) \circ T \circ \rho_V(g)^{-1}$. (This ensures the crucial fact that the subspace $\mathrm{Hom}(V, W)^G$ is precise the subspace $\mathrm{Hom}_G(V, W)$ of G-equivariant homomorphisms.)

To save notation, we shall now write $g.v$ rather than $\rho(g)(v)$. The following lemma will be used a lot.

Lemma 1.3. *Let $L : W' \to W$ be a $\mathbf{C}$-linear map between finite-dimensional $\mathbf{C}$-vector spaces and $f : G \to W'$ a continuous function. Then $L(\int_G f(g)\,\mathrm{d}g) = \int_G (L \circ f)(g)\,\mathrm{d}g$.*

In this statement $\int_G f(g)\,\mathrm{d}g$ is a "vector-valued" integral.

Proof. By computing relative to $\mathbf{C}$-bases of W and W', we reduce to the case $W = W' = \mathbf{C}$ and $L(z) = cz$ for some $c \in \mathbf{C}$. This case is obvious. ■

2. Applications

Consider the linear operator $T : V \to V$ defined by the vector-valued integral $v \mapsto \int_G g.v\,\mathrm{d}g$. In the case of finite groups, this is the usual averaging projector onto V^G. Let's see that it has the same property in general. Since $\int_G \mathrm{d}g = 1$, it is clear (e.g., by computing relative to a $\mathbf{C}$-basis of V) that $T(v) = v$ if v lies in the subspace V^G of G-invariant vectors. Moreover, T lands inside V^G: by Lemma 1.3, we may compute

$$g'.T(v) = \int_G g'.(g.v)\,\mathrm{d}g = \int_G (g'g).v\,\mathrm{d}g = \int_G g.v\,\mathrm{d}g = T(v),$$

where the second to last equality uses the invariance of the measure under left translation by g'^{-1}.

Since T is a linear projector on V^G, its trace as an endomorphism of V is $\dim(V^G)$. In terms of integration of continuous fuctions $G \to \mathrm{End}(V)$, we see that $T = \int_G \rho(g)\,\mathrm{d}g$. Thus, applying Lemma 1.3 to the linear map $\mathrm{Tr} : \mathrm{End}(V) \to \mathbf{C}$, we conclude that

$$\dim(V^G) = \mathrm{Tr}(T) = \int_G \mathrm{Tr}(\rho(g))\,\mathrm{d}g = \int_G \chi_V(g)\,\mathrm{d}g.$$

Applying this identity to the G-representation space $\mathrm{Hom}(V, W)$ mentioned earlier (with V and W any two finite-dimensional G-representations), we obtain:

Proposition 2.1.

$$\dim_{\mathbf{C}} \mathrm{Hom}_G(V, W) = \int_G \overline{\chi}_V(g)\chi_W(g)\,\mathrm{d}g.$$

Rather generally, for any continuous functions $\psi, \phi : G \rightrightarrows \mathbf{C}$ we define

$$\langle \psi, \phi \rangle = \int_G \overline{\psi}(g)\phi(g)\,\mathrm{d}g,$$

so $\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(V, W)$. Thus, by Schur's Lemma for irreducible finite-dimensional $\mathbf{C}$-linear representations of G (same proof as for finite groups), we conclude that if V and W are *irreducible* then $\langle \chi_V, \chi_W \rangle$ is equal to 1 if $V \simeq W$ and it vanishes otherwise. These are the *orthogonality relations* among the characters.

By using integration in place of averaging, any quotient map $V \twoheadrightarrow W$ between finite-dimensional continuous representations of G admits a G-equivariant section, so a general finite-dimensional continuous representation V of G over $\mathbf{C}$ decomposes up to isomorphism as a direct sum $\oplus V_j^{\oplus n_j}$ where the V_j's are the pairwise non-isomorphic irreducible subrepresentations of G inside V and n_j is the multiplicity with which it occurs (explicitly, $n_j = \dim \mathrm{Hom}_G(V_j, V)$ due to Schur's Lemma, as for finite groups). Thus, in view of the orthogonality relations,

$$\langle \chi_V, \chi_V \rangle = \sum n_j^2.$$

This yields:

Corollary 2.2. *A representation (ρ, V) of G is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$; in the reducible case this pairing is larger than 1. In particular, χ_V determines whether or not V is an irreducible representation of G.*

In fact, we can push this computation a bit further: if W is an irreducible representation of G and $W \not\simeq V_j$ for any j (i.e., W does not occur inside V) then $\langle \chi_V, \chi_W \rangle = 0$, so an irreducible representation W of G occurs inside V if and only if $\langle \chi_V, \chi_W \rangle \neq 0$, in which case this pairing is equal to the multiplicity of W inside V. Thus, writing $n_{V,W} := \langle \chi_V, \chi_W \rangle$, we have

$$V \simeq \bigoplus_{n_{V,W} \neq 0} W^{\oplus n_{V,W}}.$$

This reconstructs V from data depending solely on χ_V (and general information associated to every irreducible representation of G). In particular, it proves:

Corollary 2.3. *Every finite-dimensional continuous representation V of G is determined up to isomorphism by χ_V.*