Math 210B. Finite-dimensional commutative algebras over a field

Let A be a nonzero finite-dimensional commutative algebra over a field k. Here is a general structure theorem for such A:

Theorem 0.1. *The set* $\mathrm{Max}(A)$ *of maximal ideals of* A *is finite, all primes of* A *are maximal and minimal, and the natural map*

$$A \to \prod_{\mathfrak{m}} A_{\mathfrak{m}}$$

is an isomorphism, with each $A_{\mathfrak{m}}$ *having nilpotent maximal ideal.*

In particular, if A *is reduced then* $A \simeq \prod_{i=1}^{n} k_i$ *for fields* k_i*, with the maximal ideals given by the kernels of the projections* $A \to k_i$*.*

The assertion in the reduced case follows from the rest since if A is reduced then so is each $A_{\mathfrak{m}}$ (and hence its nilpotent unique maximal ideal vanishes, implying $A_{\mathfrak{m}}$ must be a field). Note also that the nilpotence of the maximal ideal $\mathfrak{m}A_{\mathfrak{m}}$ implies that for some large n we have

$$A_{\mathfrak{m}} = A_{\mathfrak{m}}/\mathfrak{m}^n A_{\mathfrak{m}} = (A/\mathfrak{m}^n)_{\mathfrak{m}} = A/\mathfrak{m}^n$$

(final equality since $\mathfrak{m}$ is maximal in A), so the isomorphism in the Theorem can also be expressed as saying $A \simeq \prod_{\mathfrak{m}} A/\mathfrak{m}^n$ for large n.

Most of the proof of this result is worked out in HW1 Exercise 7, and here we just address one point: the nilpotence of the maximal ideal of the local ring $A_{\mathfrak{m}}$ at each maximal ideal $\mathfrak{m}$ of A. That is, we claim that the maximal ideal $M := \mathfrak{m}A_{\mathfrak{m}}$ is nilpotent. To establish such nilpotence, note that M is finitely generated as an $A_{\mathfrak{m}}$-module since $A_{\mathfrak{m}}$ is noetherian (as A is obviously noetherian!). Thus, by Nakayama's Lemma, $M^n = 0$ if and only if $M^{n+1} = M^n$ (as when this latter equality holds then $M \cdot M^n = M^n$ with M^n a finitely generated module over the local ring $A_{\mathfrak{m}}$).

It now suffices to show that $M^n = M^{n+1}$ inside $A_{\mathfrak{m}}$ for large n, so it suffices to show that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ inside A for large n. (This latter equality would *not* imply anything about vanishing of $\mathfrak{m}^n$ in A, as we know it cannot, since A is not generally local and hence Nakayama's Lemma is not applicable.) So we shall prove $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for all large n. The sequence of ideals $\{\mathfrak{m}^n\}_{n\geq 1}$ in A is a decending chain of k-subspaces of the *finite-dimensional* k-vector space A. But any such descending chain must have dimensions eventually stabilizing, so then the chain itself must stabilize for dimension reasons. Thus, we get the desired equality for large n.