

MATH 210B. ABELIAN DUAL GROUP AND FINITE FOURIER TRANSFORM

Let G be a finite abelian group, and $\widehat{G} = \text{Hom}(G, \mathbf{C}^\times)$ its finite dual group. We identify the group algebra $\mathbf{C}[G]$ with the set of functions $f : G \rightarrow \mathbf{C}$, assigning to f the element $\sum_{g \in G} f(g)[g]$ (so $[g]$ corresponds to the “point mass” at g ; i.e., the function with value 1 at g and 0 elsewhere). Recall that this identifies multiplication in $\mathbf{C}[G]$ with the convolution product

$$(f_1 * f_2)(g) = \sum_{h \in G} f_1(h)f_2(h^{-1}g).$$

For any $f : G \rightarrow \mathbf{C}$, we define its *Fourier transform* $\widehat{f} : \widehat{G} \rightarrow \mathbf{C}$ to be the function $\widehat{f} : \chi \mapsto \sum_{g \in G} f(g)\chi(g)$. The aim of this handout is to explain the representation-theoretic significance of this operation in terms of $\mathbf{C}[G]$ viewed as the regular representation of G , and in particular to use this and the double-duality of finite abelian groups (i.e., the ability to view G canonically as the dual of \widehat{G}) to deduce some familiar-looking properties of the Fourier transform with no messy calculations. The extension of these ideas to a general finite groups G involves more serious input from representation theory to be discussed later.

1. DOUBLE DUALITY

As a preliminary step, we explain for k algebraically closed of characteristic 0 how the relationship between G and $\widehat{G} = \text{Hom}(G, k^\times)$ is rather “symmetric” in nature, as with the relationship between a finite-dimensional vector space and its dual over a field. We saw in class that if G is cyclic of order n then \widehat{G} is cyclic of order n . Here is an extension of that fact to general finite abelian G :

Theorem 1.1. *Non-canonically, \widehat{G} is isomorphic to G . Moreover, the bi-multiplicative pairing $G \times \widehat{G} \rightarrow k$ defined by $(g, \chi) \mapsto \chi(g)$ is perfect in the sense that the map $G \rightarrow G^{\wedge\wedge}$ defined by $g \mapsto (\text{ev}_g : \chi \mapsto \chi(g))$ is an isomorphism.*

Proof. The case of cyclic G was discussed in class. To bootstrap from that to the general case, we note that G is isomorphic to a direct product $C_1 \times \cdots \times C_r$ of finitely many cyclic groups, so the key issue is to check that the formation of the dual group of G and of the pairing between G and \widehat{G} are each naturally compatible with direct products in G .

Firstly, for finite abelian G_1 and G_2 we have a natural homomorphism

$$\alpha : \widehat{G}_1 \oplus \widehat{G}_2 \rightarrow (G_1 \oplus G_2)^\wedge$$

defined by $(\chi_1, \chi_2) \mapsto ((g_1, g_2) \mapsto \chi_1(g_1)\chi_2(g_2))$. We claim that this is an isomorphism, in which case it follows by induction on the number of cyclic groups used to obtain G as a direct product that G and \widehat{G} are always non-canonically isomorphic. To verify this isomorphism property is an elementary exercise, since any homomorphism $G_1 \oplus G_2 \rightarrow k^\times$ is determined by its restriction to each factor and that from a given pair of homomorphisms $\chi_i : G_i \rightarrow k^\times$ we can make $\chi : G_1 \oplus G_2 \rightarrow k^\times$ recovering χ_i as its G_i -restriction via the recipe in the definition of α .

As for the compatibility of the pairing with respect to direct products in G , going through definitions shows that the diagram of pairings

$$\begin{array}{ccc} (G_1 \oplus G_2) \times (\widehat{G}_1 \oplus \widehat{G}_2) & \xlongequal{\quad} & (G_1 \times \widehat{G}_1) \times (G_2 \times \widehat{G}_2) \longrightarrow k^\times \times k^\times \\ \downarrow 1 \times \alpha & & \downarrow \text{mult} \\ (G_1 \oplus G_2) \times (G_1 \oplus G_2)^\wedge & \xrightarrow{\quad} & k \end{array}$$

commutes. It then follows (check!) that the double-duality map

$$(G_1 \oplus G_2) \rightarrow (G_1 \oplus G_2)^{\wedge\wedge} \simeq (G_1^\wedge \oplus G_2^\wedge)^\wedge = G_1^{\wedge\wedge} \oplus G_2^{\wedge\wedge}$$

for $G_1 \oplus G_2$ is the direct sum of the ones for G_1 and G_2 . Hence, the isomorphism property of this map for $G_1 \oplus G_2$ reduces to the same for G_1 and G_2 separately. In this way, the isomorphism property for the double-duality map in general reduces to the case of cyclic groups that was discussed in class. ■

2. IDEMPOTENTS AND ISOTYPIC PROJECTORS

From now on, $k = \mathbf{C}$. Let $V = \mathbf{C}[G]$ be the regular representation of G . This has dimension $\#G = \#\widehat{G}$, and we know that each of the $\#\widehat{G}$ irreducible (even 1-dimensional) representations of G occurs in V . Thus, for dimension reasons, each isotypic space in V is a line. That is, $V = \bigoplus_{\chi} L_{\chi}$ where L_{χ} is a line on which G acts through multiplication against $\chi : G \rightarrow \mathbf{C}^{\times}$. In fact, there is an interesting explicit basis vector for L_{χ} , given by

$$e_{\chi} = (1/\#G) \sum_{g \in G} \chi(g^{-1})[g] \neq 0;$$

indeed, for all $g_0 \in G$ we compute

$$g_0 \cdot e_{\chi} = (1/\#G) \sum_{g \in G} \chi(g^{-1})[g_0g] = (1/\#G) \sum_{h \in G} \chi(h^{-1}g_0)[h] = \chi(g_0)e_{\chi}.$$

A notable feature of the e_{χ} 's is that they are pairwise orthogonal idempotents in $\mathbf{C}[G]$; that is, $e_{\chi}^2 = e_{\chi}$ and $e_{\chi}e_{\psi} = 0$ when $\psi \neq \chi$. To verify these properties, we require a very useful yet elementary pair of identities:

Lemma 2.1. *For any $g \neq 1$, $\sum_{\chi} \chi(g) = 0$. Likewise, for any $\chi \neq 1$, $\sum_g \chi(g) = 0$.*

Proof. The two statements are equivalent to each other via double-duality; that is, if we can prove one of them in general then the other is exactly the same assertion with the roles of G and \widehat{G} swapped. We focus on the second statement: if $S := \sum_g \chi(g)$ then for any $g_0 \in G$ we have (by ‘‘change of variables’’)

$$\chi(g_0)S = \sum_g \chi(g_0g) = \sum_g \chi(g) = S,$$

so $(\chi(g_0) - 1)S = 0$. Since $\chi \neq 1$, we can choose g_0 so that $\chi(g_0) \neq 1$. Hence, $S = 0$. ■

To use Lemma 2.1, we first note that for any $\chi, \psi \in \widehat{G}$,

$$e_{\chi}e_{\psi} = (1/\#G)^2 \sum_{g,h \in G} \chi(g^{-1})\psi(h^{-1})[gh] = (1/\#G)^2 \sum_{g,h \in G} \chi((gh)^{-1})\chi(h)\psi(h^{-1})[gh].$$

Making a change of variable $(g, h) \mapsto (gh, h)$, the sum becomes

$$\sum_{g,h} \chi(g^{-1})\chi(h)\psi(h^{-1})[g] = \sum_g \left(\sum_h (\chi/\psi)(h) \right) \chi(g^{-1})[g].$$

This vanishes if $\chi/\psi \neq 1$ (i.e., if $\chi \neq \psi$) by Lemma 2.1, and otherwise it equals $\#G \sum_g \chi(g^{-1})[g] = (\#G)^2 e_{\chi}$. Hence, the idempotent and pairwise orthogonal properties of the e_{χ} 's is established. Let's summarize these in the following version that will adapt to non-abelian G later.

Proposition 2.2. *The \mathbf{C} -linear map $\mathbf{C}[G] \rightarrow \mathbf{C}^{\widehat{G}} := \prod_{\chi} \mathbf{C}$ defined by $[g] \mapsto (\chi(g))_{\chi}$ is an isomorphism of commutative \mathbf{C} -algebras, carrying e_{χ} to the idempotent in $\mathbf{C}^{\widehat{G}}$ whose χ -component is 1 and whose other components vanish.*

For any G -representation W viewed as a $\mathbf{C}[G]$ -module, since the idempotents e_{χ} are pairwise orthogonal and sum to 1 it follows that the images $W_{\chi} := e_{\chi}W$ are subrepresentations that directly span W ; i.e., $W = \bigoplus_{\chi} W_{\chi}$. Indeed, since $1 = \sum_{\chi} e_{\chi}$ certainly $\sum_{\chi} W_{\chi} = W$, and this is a direct sum because applying e_{ψ} to W kills W_{χ} for $\chi \neq \psi$ and acts as the identity on elements of W_{ψ} (so if $w_{\chi} \in W_{\chi}$ and $\sum_{\chi} w_{\chi} = 0$ then applying any $e_{\psi} \in \mathbf{C}[G]$ to this yields $w_{\psi} = 0$ for any ψ).

What is the meaning of this direct sum decomposition of W ? It is exactly the isotypic decomposition, with W_{χ} the χ -isotypic subspace (so since e_{χ} kills W_{ψ} for $\psi \neq \chi$ and acts as the identity on W_{χ} , the operator $e_{\chi} \in \mathbf{C}[G]$ acting on W is exactly ‘‘projection to the χ -isotypic subspace’’). Indeed, we just have to check that any $[g] \in \mathbf{C}[G]$ acts on W_{χ} as multiplication by $\chi(g)$, but by definition of W_{χ} as $e_{\chi}W$ we see that it suffices to check the identity $[g]e_{\chi} = \chi(g)e_{\chi}$ in $\mathbf{C}[G]$ which we verified near the start.

3. THE TRANSFORM

Since the e_χ 's are bases for the lines L_χ that directly span the regular representation V , we see that every $f : G \rightarrow \mathbf{C}$ viewed as an element of $\mathbf{C}[G] = V$ admits a unique expansion

$$f = \sum_{\chi} c_{\chi,f} e_{\chi}$$

for $c_{\chi,f} \in \mathbf{C}$. This is just the isotypic decomposition of f viewed as a vector in V . Its coefficients are *exactly* the Fourier transform:

Lemma 3.1. *For each $\chi \in \widehat{G}$, $c_{\chi,f} = \widehat{f}(\chi)$. In other words, $f = \sum_{\chi} \widehat{f}(\chi) e_{\chi}$.*

Proof. By \mathbf{C} -linearity of the Fourier transform,

$$\widehat{f} = \sum_{\chi} c_{\chi,f} (e_{\chi})^{\wedge}$$

as functions on \widehat{G} . But what is the Fourier transform $(e_{\chi})^{\wedge}$ of the function $e_{\chi} \in \mathbf{C}[G]$ on G ? I claim that this transform, as a function on \widehat{G} , is the ‘‘point mass’’ supported at χ ; in terms of $\mathbf{C}[\widehat{G}]$ this says it is equal to $[\chi]$. This assertion is an immediate calculation from the definition of $e_{\chi} = (1/\#G) \sum_g \chi(g^{-1})[g]$:

$$(e_{\chi})^{\wedge}(\psi) = \sum_g e_{\chi}(g) \psi(g) = \sum_g (1/\#G) \chi(g^{-1}) \psi(g) = (1/\#G) \sum_g (\psi/\chi)(g),$$

and by Lemma 2.1 this vanishes for $\psi \neq \chi$ and equals 1 for $\psi = \chi$. Thus, $\widehat{f} = \sum_{\chi} c_{\chi,f} [\chi]$, so by definition of the dictionary between functions on \widehat{G} and elements of $\mathbf{C}[\widehat{G}]$ this forces $c_{\chi,f} = \widehat{f}(\chi)$ as desired. \blacksquare

Having given the interpretation of \widehat{f} in terms of the expansion of f relative to the basis of e_{χ} 's for the isotypic subspaces of V , some expected properties of the transform now drop out easily. First, since multiplication in the group algebra corresponds to convolution of functions $G \rightarrow \mathbf{C}$, we see that $(f_1 * f_2)^{\wedge}(\chi)$ is the e_{χ} -coefficient for the product

$$\left(\sum_{\psi} \widehat{f}_1(\psi) e_{\psi} \right) \left(\sum_{\theta} \widehat{f}_2(\theta) e_{\theta} \right)$$

in $\mathbf{C}[G]$. But the e_{ψ} 's are pairwise orthogonal idempotents, so this product is equal to $\sum_{\chi} \widehat{f}_1(\chi) \widehat{f}_2(\chi) e_{\chi}$. Hence, $(f_1 * f_2)^{\wedge} = \widehat{f}_1 \cdot \widehat{f}_2$ (reminiscent of the classical Fourier transform).

The Fourier inversion formula requires some care with factors of $\#G$. To determine the Fourier transform of \widehat{f} , we want to read off the coefficients when $\widehat{f} \in \mathbf{C}[\widehat{G}]$ is expanded in the basis of idempotents

$$e_g = (1/\#\widehat{G}) \sum_{\chi} \chi(g^{-1})[\chi].$$

We claim that

$$\widehat{f} = \sum_{g \in G} \#G \cdot f(g^{-1}) e_g,$$

from which it would follow by Lemma 3.1 applied with the roles of G and \widehat{G} swapped (!) that $f^{\wedge\wedge}(g) = \#G \cdot f(g^{-1})$, or in other words the inversion formula

$$(1) \quad f = (1/\#G) f^{\wedge\wedge}(g^{-1})$$

that is reminiscent of the classical Fourier inversion formula.

To establish the asserted formula for \widehat{f} when expanded in the basis of idempotents $\{e_g\}$ for $\mathbf{C}[\widehat{G}]$, we simply plug in the definition of e_g (as sum over χ 's) to get that the right side is equal to

$$\sum_{\chi, g} f(g^{-1}) \chi(g^{-1})[\chi] = \sum_{\chi} \widehat{f}(\chi) [\chi]$$

by definition of \widehat{f} , and this is the element of $\mathbf{C}[\widehat{G}]$ corresponding to the function $\widehat{f} : \widehat{G} \rightarrow \mathbf{C}$.

Remark 3.2. The annoying factor $1/\#G$ in the Fourier inversion formula (1) is analogous to the annoying factor of $1/2\pi$ that multiplies against various integrals in classical Fourier analysis. These factors are best understood, and then artfully avoided, from the viewpoint of dual measures in the wider context of Pontryagin duality for locally compact Hausdorff topological abelian groups. That is a story for another day, but here is the analogous renormalization on Euclidean space: by using the convention to define the Fourier transform on $L^1(\mathbf{R}^n)$ to be $\widehat{f}(w) = \int_{\mathbf{R}^n} f(v)e^{2\pi i(v \cdot w)} dv$, rather than the “physics” convention with $e^{i(v \cdot w)}$ inside the integral, the scaling against powers of 2π in the formula for the inverse Fourier transform disappears.