Math 210B. Geometrization of commutative algebra and differentials

This handout explains how we can enhance the topological space $\operatorname{Spec} A$ so as to view an arbitrary commutative ring A in a manner similar to smooth functions on a smooth manifold, and also view an arbitrary A-module M in a manner similar to a vector bundle on a smooth manifold. The special case $M = \Omega^1_{B/A}$ underlies the construction of a version of deRham theory for smooth varieties, as we discuss at the end.

This circle of ideas is the starting point for modern algebraic geometry as revolutionized by Grothendieck, wherein the ability to intepret commutative algebra notions in geometric terms enables one to define "cohomological invariants" of modules inspired by ideas from algebraic topology and to study these invariants by means of structures analogous to those seen in topology: excision sequences, cup products, and duality theorems.

1. Some function theory

In class we saw how to visualize $\operatorname{Spec}(k[t_1,\dots,t_n])$ for $k = \overline{k}$ as an upgrading of $\operatorname{MaxSpec}(k[t_1,\dots,t_n])$ in which a new generic point is inserted for every classical irreducible closed subset that isn't a point. This helps us to visualize $\operatorname{Spec} A$ in general. We also saw that an element $a \in A$ defines a "function" on $X = \operatorname{Spec} A$ by defining the *value* $a(x)$ at $x = \mathfrak{p} \in X$ to be $a \bmod \mathfrak{p} \in \operatorname{Frac}(A/\mathfrak{p}) =: \kappa(x)$. In the classical case with x a closed point (so naturally $k = \kappa(x)$) we saw that this recovers the classical notion of evaluation of a polynomial at a point of k^n. But this viewpoint (apart from the technical subtlety that as we vary x the "values" $a(x)$ are lying in fields $\kappa(x)$ that can vary all over the place) loses all information about nilpotents in A since $a(x) = 0$ in $\kappa(x)$ for nilpotent a.

Consideration of the non-transverse intersection of $y = x^2$ and the x-axis at the origin showed us that there is geometric merit in allowing non-zero nilpotent "functions". So we seek a way to interpret A as a "ring of functions" on $X = \operatorname{Spec} A$ in a manner that is more refined than the operation $a \mapsto (x \mapsto a(x))_{x\in X}$. In class we introduced a way to do this, by associating to every basic affine open subset $U = X_a \subset X = \operatorname{Spec} A$ the ring $\mathscr{O}(U) := S_U^{-1}A = A_a$ where S_U is the multiplicative set of elements $a' \in A$ that are "units on U"; i.e., $a'(x) \neq 0$ for all $x \in U$ (or equivalently, $a' \notin \mathfrak{p}$ for all primes $\mathfrak{p}$ corresponding to points of U). We saw how to define "restriction maps"

$$\rho^U_V : \mathscr{O}(U) \to \mathscr{O}(V)$$

(usually denoted $f \mapsto f|_V$) for any inclusion $V \subset U$ of basic affine open subsets of X (uniquely determined as A-algebra maps), and our goal is to show that this assignment of rings to (certain!) open sets in X satisfies a gluing property similar to that of gluing smooth functions on manifolds.

To make a fully satisfactory analogue of the assignment $W \mapsto C^\infty(W)$ for open subsets W in a smooth manifold Z, we should really define a suitable ring $\mathscr{O}(U)$ for *every* open subset $U \subset X$ and appropriate "restriction maps" $\rho^U_V : \mathscr{O}(U) \to \mathscr{O}(V)$ whenever $V \subset U$, but that entails a long technical digression. The real content lies in the case of basic affine open sets and *finite* coverings of them by other basic affine open sets. More specifically, the crux of the matter is the following result:

Theorem 1.1. *Let $\{U_1,\dots,U_n\}$ be a finite collection of basic affine open sets $U_i = X_{a_i}$ that cover $X = \operatorname{Spec} A$. The natural map of rings*

$$A = \mathscr{O}(X) \to \{(f_i) \in \prod \mathscr{O}(U_i) \mid f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j} \text{ in } \mathscr{O}(U_i \cap U_j) \text{ for all } i,j\}$$

is an isomorphism. Equivalently, the natural map of rings

$$A \to \{(f_i) \in \prod A_{a_i} \mid f_i = f_j \text{ in } A_{a_i a_j} \text{ for all } i,j\}$$

is an isomorphism.

The injectivity of the map in question is analogous to the obvious property that a smooth function on a manifold Z is determined by its restriction to each member of an open cover $\{W_i\}$ of Z, and the surjectivity is analogous to the fact that smooth functions h_i on the W_i's that agree on the overlaps $W_i \cap W_j$ arise via restriction to the W_i's of a common (uniquely determined) smooth function h on Z.

Proof. The condition that the X_{a_i}'s cover X is exactly that the ideal (a_i) is the unit ideal, as we saw in class, so $\sum a_i' a_i = 1$ for some $a_i' \in A$. This will serve the role of a "partition of unity" for making a global construction. The easier part of the bijectivity assertion is injectivity, so we first dispose of that.

For $a, a' \in A$ with the same image in every A_{a_i}, we wish to show that $a = a'$ in A. This has content because, in contrast with the case of domains, the maps $A \to A_{a_i}$ may all fail to be injective. (For example, if $A = k[x,y]/(xy(y-1))$ then $\operatorname{Spec} A$ is visualized as the union of the coordinate axes and the line $y = 1$, and we can take $\{a_i\}$ to be $\{y, y-1\}$ with $A \to A_y$ killing the nonzero $x(y-1) \in A$ and $A \to A_{y-1}$ killing the nonzero $xy \in A$.)

By passing to the difference $a' - a$, our problem is to show that if $a \in A$ has vanishing image in every A_{a_i} then $a = 0$ in A. Vanishing in A_{a_i} means that $a_i^{n_i} a = 0$ in A for some $n_i \geq 0$, so for $n = \max_i n_i$ we have $a_i^n a = 0$ in A for all i. In other words, a is annihilated by every a_i^n. But we have a relation $\sum a_i' a_i = 1$, so by raising it to an enormous power (depending on n and the number of i's) we see that likewise the a_i^n's generated 1. Multiplying such a relation $\sum b_i a_i^n = 1$ against a then gives $0 = a$ in A as desired.

Now we turn to surjectivity, which lies deeper. Even if A is a domain this part is not obvious, since elements in $\operatorname{Frac}(A)$ do not have a well-defined "reduced form" expression when A is not a UFD, so we cannot do naive denominator-chasing in general. (That is, even for a domain A, it is not obvious that the intersection $\bigcap_{\mathfrak{m}} A_{\mathfrak{m}}$ inside $\operatorname{Frac}(A)$ coincides with A.) The argument will be an artful replacement of the idea of chasing denominators, inspired by the use of partitions of unity in differential geometry. We can write every $f_i \in A_{a_i}$ in the form $x_i/a_i^{n_i}$ for an integer $n_i \geq 0$, and we can rewrite these fractions to be of the form y_i/a_i^n for a common exponent n. The equality of the images of f_i and f_j in $A_{a_i a_j}$ says that some $(a_i a_j)^{N_{ij}}(a_j^n y_i - a_i^n y_j)$ vanishes in A with an integer $N_{ij} \geq 0$, and we can certainly increase the N_{ij}'s to all be the same and positive. That is, for integers $n \geq 0$ and $m > 0$ we have

$$(a_i a_j)^m (a_j^n y_i - a_i^n y_j) = 0$$

in A. That is, $a_j^{n+m}(a_i^m y_i) = a_i^{n+m}(a_j^m y_j)$ for all i, j.

Since $f_i = y_i/a_i^n = (a_i^m y_i)/a_i^{n+m}$ and $X_{a_i} = X_{a_i^{n+m}}$ (as $n + m > 0$), we can replace y_i with $a_i^m y_i$ and a_i with a_i^{n+m} to get to the situation that $f_i = y_i/a_i$ for all i and $a_j y_i = a_i y_j$ in A for all i, j. (This is the naive equality in A we would get by "cross multiplying" in the formal equality of fractions $y_i/a_i = y_j/a_j$. But keep in mind that we have replaced the original a_i's with suitable powers.) Now comes the partition-of-unity step: the a_i's generate 1 since the X_{a_i}'s cover X, so we have $1 = \sum a_i' a_i$ for some $a_i' \in A$. Thus, multiplying both sides by y_j gives

$$y_j = \sum a_i' a_i y_j = \sum a_i' y_i a_j = a_j \cdot \sum a_i' y_i,$$

so in A_{a_j} we have $f_j = y_j/a_j = \sum a_i' y_i$. In other words, the element $a = \sum a_i' y_i \in A$ maps to f_j in A_{a_j} for every j. ■

2. Modules

The preceding methods adapt to a general A-module M (going beyond the case $M = A$), as follows. For any $U = X_a \subset X := \operatorname{Spec}(A)$, we claim that there is a unique A-linear isomorphism $S_U^{-1}M \simeq M_a$ respecting the natural maps from M to both sides, where (as above) S_U is the set of $a' \in A$ non-vanishing at all $u \in U$. (Note $0 \in S_U$ if $U = \emptyset$.) Since $a \in S_U$ certainly by the universal property of localization there is a unique A-linear map $M_a \to S_U^{-1}M$ respecting the natural maps from M to both sides. But M_a is a module over $A_a = S_U^{-1}A$, so all elements of S_U act invertibly on M_a. Hence, the natural map $M \to M_a$ factors uniquely through $S_U^{-1}M$ as an A-linear (or equivalently A_a-linear) map, and by chasing numerators from M via A-linearity we see that the two maps we have constructions $S_U^{-1}M \to M_a$ and $M_a \to S_U^{-1}M$ are inverse to each other.

Definition 2.1. For each U (even $U = \emptyset$!), define $\widetilde{M}(U) := S_U^{-1}M$.

The reason we work with S_U-localization rather than a-localization is to convey the precise sense in which this really only depends on the open subset $U \subset X$ and *not* on the specific (non-canonical) element $a \in A$ for which $X_a = U$.

For any open subset $U' = X_{a'} \subset U$ there is a unique $\rho^U_{U'} : \widetilde{M}(U) \to \widetilde{M}(U')$ (denoted $m \mapsto m|_{U'}$) linear over $\mathscr{O}(U) \to \mathscr{O}(U')$ and respecting the maps from M to both sides. Indeed, this says exactly that there exists a unique map $S_U^{-1}M \to S_{U'}^{-1}M$ linear over $S_U^{-1}A \to S_{U'}^{-1}A$ respecting the map from M to both sides, and by the universal property of S_U-localization this says exactly that the elements of S_U act invertibly on $S_{U'}^{-1}M$. But this latter property is obvious because $S_U \subset S_{U'}$ (since $U' \subset U$, so any $a \in A$ non-vanishing at all points of U is certainly non-vanishing at all points of U').

This construction satisfies transitivity with respect to inclusions of basic affine opens $U'' \subset U' \subset U$ in the sense that the composite map

$$\widetilde{M}(U) \to \widetilde{M}(U') \to \widetilde{M}(U'')$$

(reminiscent of transitivity for restriction of vector fields on open subsets of a smooth manifold Z, expressed in terms of the assignment to each open $W \subset Z$ the $C^\infty(W)$-module $\mathrm{Vec}_Z(W)$ of smooth vector fields on W). This transitivity property concerns identifying the composite map

$$S_U^{-1}M \to S_{U'}^{-1}M \to S_{U''}^{-1}M$$

with $\rho^U_{U''}$, but this is clear because A-linearity of all constructions under consideration reduces this to identifying the composite map $M \to S_{U''}^{-1}M$, which in turn is clear by numerator-chasing in the constructions.

Here is the analogue of Theorem 1.1:

Theorem 2.2. *Let $\{U_1, \dots, U_n\}$ be a finite cover of X by basic affine open $U_i = X_{a_i}$. Prove the following natural A-linear map is an isomorphism:*

$$M = \widetilde{M}(X) \to \{(m_i) \in \prod \widetilde{M}(U_i) \mid \ m_i|_{U_i \cap U_j} = m_j|_{U_i \cap U_j} \text{ in } \widetilde{M}(U_i \cap U_j) \text{ for all } i, j\}.$$

Remark 2.3. We allow some $U_i = \emptyset$, and more importantly some $U_i \cap U_j = \emptyset$, as will certainly happen in practice).

The proof of Theorem 2.2 is literally *identical* to the proof of Theorem 1.1, since all we used about the $\mathscr{O}(U)$'s is not that each $S_U^{-1}A$ is the S_U-localization of a fixed ring but rather is an S_U-localization of a fixed A-module. By working with numerators in M rather than in A, the previous argument carries over with no changes whatsoever.

As a final step in this process of expressing rings and modules in somewhat geometric terms (or at least more vividly "local" terms), we consider the relation to the construction of germs of functions in differential geometry. If Z is a smooth manifold and $z \in Z$ is a point, we recall that the *ring of germs* $\mathscr{C}^\infty_{Z,z}$ of smooth functions at x is the ring

$$\varinjlim_{z \in W} C^\infty(W)$$

consisting of smooth functions h defined near z taken up to the equivalence relation $h_1 \sim h_2$ if h_1 coincides with h_2 near z. This is a local ring since a smooth function near z is invertible near z if and only if its value at z (and hence at all points near z!) is non-vanishing). Likewise, we can define the $\mathscr{C}^\infty_{Z,z}$-module

$$\mathrm{Vec}_{Z,z} = \varinjlim_{z \in W} \mathrm{Vec}_Z(W)$$

of germs of smooth vector fields near z; this is a free module of finite rank, a basis given by the classes of $\partial_{t_1}, \dots, \partial_{t_n}$ for local coordintes $\{t_1, \dots, t_n\}$ near z (a "local frame" for vector fields near z).

We can make similar constructions in the context of the preceding commutative algebra setting: for $x = \mathfrak{p} \in X$, the set of basic affine open subsets $U \subset X$ containing x is directed by reverse inclusion, so we may define the ring

$$\mathscr{O}_x := \varinjlim_{x \in U} \mathscr{O}(U)$$

where the direct limit is taken with respect to the (directed system of!) "restriction maps" as made above. This is a local ring: there is an evident map $f : \mathscr{O}_x \to \kappa(\mathfrak{p})$ into the residue field at $x = \mathfrak{p}$ (coming from passage to the direct limit on the compatible maps $A_a \to A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ for all $a \in A - \mathfrak{p}$), and $\ker f$ is the unique maximal ideal because for any $a \notin \mathfrak{p}$ and fraction $a'/a^n \in A_a$ whose image in $A_\mathfrak{p}$ is not in $\ker f$ must satisfy $a' \notin \mathfrak{p}$ (why?), thereby providing $a^n/a' \in A_{a'}$ whose class in $\mathscr{O}_x$ is a reciprocal of the class of a'/a^n.

We can also define $\widetilde{M}_x := \varinjlim_{x\in U} \widetilde{M}(U)$ that is naturally an $\mathscr{O}_x$-module, and the connection to commutative algebra is:

Proposition 2.4. *There is a unique A-algebra isomorphism $A_{\mathfrak{p}} \simeq \mathscr{O}_x$, and over this ring isomorphism is a unique module isomorphism $M_{\mathfrak{p}} \simeq \widetilde{M}_x$ respecting the maps from M to both sides.*

This result explains how "localization at $\mathfrak{p}$" on rings and modules amounts to working locally at an actual point in a manifold.

Proof. We have seen that the units in $\mathscr{O}_{X,x}$ are precisely the elements with nonzero image under the natural map $f : \mathscr{O}_{X,x} \to \kappa(\mathfrak{p})$. Moreover, for an element $a \in A$, its image in $\mathscr{O}_{X,x}$ is carried by f to the class of $a \bmod \mathfrak{p} \in \mathrm{Frac}(A/\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Thus, a maps to a unit in $\mathscr{O}_{X,x}$ if and only if $a \in A - \mathfrak{p}$, so in particular the natural ring map $A \to \mathscr{O}_{X,x}$ uniquely factors through a ring map

$$\varphi : A_{\mathfrak{p}} \to \mathscr{O}_x.$$

By definition of the right side as a direct limit, every element in $\mathscr{O}_x$ arises from some A_a with $a \notin \mathfrak{p}$ and so arises from $A_{\mathfrak{p}}$ via the unique A-algebra map $A_a \to A_{\mathfrak{p}}$ (by unraveling definitions, or checking equalities of maps out of localizations of A by chasing numerators and units). Thus, φ is surjective.

To complete the proof that φ is an isomorphism, we have to prove that it is injective. Any element ξ in the kernel can be written as a fraction coming from some A_a with $a \in A - \mathfrak{p}$, say in the form a'/a^n without loss of generality. The condition of vanishing in the direct limit $\mathscr{O}_x$ implies vanishing at some stage of the limit process, which is to say that there is a basic affine open neighborhood $U = X_b$ of x (i.e., $b \notin \mathfrak{p}$) such that $U \subset X_a$ and $a'/a^n \in \mathscr{O}(X_a)$ restricts to zero on U. But this exactly says that the restriction map $\rho : A_a \to A_b$ (arising from the inclusion of X_b inside X_a) kills a'/a^n. Since $A_a \to A_{\mathfrak{p}}$ carries a'/a^n to ξ by design, it remains to observe that this latter ring map factors as the composition of ρ and the natural map $A_b \to A_{\mathfrak{p}}$ (since we can compare any two maps from A_a to a ring by comparing after composing back to A, due to the universal property of localization).

Having established that φ is an isomorphism, the exact same argument works with $\widetilde{M}$ in place of $\mathscr{O}$, since A-linearity was the essential feature for the isomorphism aspect of the argument. That is, we build the desired linear map $M_{\mathfrak{p}} \to \widetilde{M}_x$ and establish its uniqueness via the universal property of localization for modules, and then prove it is an isomorphism via exactly the same arguments we went through with φ. ■

3. Smooth varieties and algebraic deRham theory

Let $A \to B$ be a map of rings, and let M be the B-module $\Omega^1_{B/A}$. On the topological space $\mathrm{Spec}(A)$ we have constructed the assignment $\mathscr{O}_A$ of rings to basic affine open sets such that the "unique gluing property" holds, and likewise for $\mathscr{O}_B$ on the topological space $\mathrm{Spec}(B)$. The pairs of data $X = (\mathrm{Spec}(B), \mathscr{O}_B)$ and $Y = (\mathrm{Spec}(A), \mathscr{O}_A)$ are (essentially) examples of affine schemes. The associated construction $\widetilde{M}$ over X is denoted $\Omega^1_{X/Y}$, called the *sheaf of relative* 1*-forms* on X over Y; this takes value $\Omega^1_{B_b/A}$ on the open set $X_b \subset X$ for any $b \in B$, and for those familiar with differential geometry this is analogous to the "relative cotangent bundle" $\mathrm{T}^*(M)/f^*(\mathrm{T}^*(N))$ for a submersion of manifolds $f : M \to N$ (the fiber of this vector bundle at any $m \in M$ is dual to the tangent space at m on the fiber $f^{-1}(f(m))$.

Example 3.1. Suppose A is a domain finitely generated algebra over a field k, and $d = \dim A$. One way to define what it means for A to be *k-smooth* is that the finitely generated A-module $\Omega^1_{A/k}$ is a "rank-d vector bundle" in the sense that it is Zariski-locally free of rank d: there is a finite cover of $\mathrm{Spec}(A)$ by basic affine open sets $\mathrm{Spec}(A_{a_i})$ such that $(\Omega^1_{A/k})_{a_i} = \Omega^1_{A_{a_i}/k}$ is a free A_{a_i}-module of rank d. When k-smoothness holds, $\Omega^1_{A/k}$ underlies a definition of *cotangent bundle* for A over k.

Grothendieck proved that this concrete definition of k-smoothness is equivalent to several other appealing notions, one of which is that if $A = k[T_1, \dots, T_n]/(f_1, \dots, f_m)$ and $J := (\partial f_i/\partial T_j) \in \mathrm{Mat}_{m\times n}(A)$ then for all $x \in \mathrm{Spec}(A)$ the matrix $J(x) \in \mathrm{Mat}_{m\times n}(\kappa(x))$ has rank d (equivalently with just closed points x). The notion of k-smoothness is closely related to the general concept of *regularity* of noetherian local rings (discussed near the end of the final Chapter of the Atiyah-MacDonald text) that makes *no reference* to an algebra structure over a field: if A is k-smooth then $A_{\mathfrak{m}}$ is regular for all maximal ideals $\mathfrak{m}$ of A, and with some hard work in commutative algebra involving *completions* it can be shown that the converse holds when k is perfect but fails otherwise (e.g., if $\operatorname{char}(k) = p > 0$ and $a \in k - k^p$ then for any $m > 1$ not divisible by p, $A := k[x,y]/(y^m - (x^p - a))$ is Dedekind and consequently regular at every maximal ideal but is not k-smooth).

Define $\Omega^p_{B/A}$ to be the pth exterior power of the B-module $\Omega^1_{B/A}$, and let $\Omega^p_{X/Y}$ be the associated object on the topological space $\operatorname{Spec}(B)$ (assigning to each open set X_b the B_b-module $\wedge^p(\Omega^1_{B_b/A}) = (\Omega^p_{B/A})_b$). In the comparison with a submersion of smooth manifolds $f : M \to N$, this is analogous to a vector bundle on M whose restriction to each fiber $f^{-1}(n)$ is the bundle of smooth p-forms on that fiber.

With some work to establish well-definedness, one can build a *de Rham complex* of A-modules

$$B \stackrel{\mathrm{d}_{B/A}}{\to} \Omega^1_{B/A} \to \Omega^2_{B/A} \to \dots$$

satisfying the usual rules (A-linear but not B-linear!), and then due to compatibility with localization this upgrades to an analogous construction

$$\mathscr{O}_B \to \Omega^1_{X/Y} \to \Omega^2_{X/Y} \to \dots$$

over the space $\operatorname{Spec}(B)$, linear over $\mathscr{O}_A$ in a suitable sense. This is called the *relative de Rham complex* for X over Y; see [EGA IV$_4$, §16.5] (especially §16.5–§16.8, noting in particular 16.6.2) for details on this algebraic construction which is made without any reliance on "local coordinates" (since in algebraic geometry one doesn't have local coordinates in a manner quite like with manifolds) and gives a novel coordinate-free way to construct the classical deRham complex for smooth (or real-analytic or complex-analytic) manifolds. (To prove *properties* of the classical deRham complex one has to use local coordinates, but the point is that making the initial construction does not require any use of considerations with coordinates.)

For smooth varieties over $\mathbf{C}$, there are deep connections between this algebraic deRham theory and the corresponding analytic deRham theory on the associated complex manifold. Since we have not discussed non-affine varieties in this course, we limit ourselves to remarking that (i) Serre proved a precise sense in which projective complex manifolds and their related deRham cohomology are algebraic in a functorial manner, (ii) Deligne and Illusie proved theorems about the Hodge cohomology of "algebraic" compact complex manifolds via *characteristic p methods* (!), and (iii) Grothendieck proved the following spectacular theorem in the affine setting by using Hironaka's resolution of singularities and the results of Serre alluded to in (i) above.

Theorem 3.2 (Grothendieck)**.** *Let $f_1, \dots, f_m \in \mathbf{C}[T_1, \dots, T_n]$ be a collection of polynomials such that their common zero locus $X \subset \mathbf{C}^n$ has all irreducible components of dimension d. Assume X is "smooth" in the sense that the Jacobian matrix $(\partial f_i/\partial T_j)$ has rank d at all points of X.*

Then for all $j \geq 0$, the jth deRham cohomology of the topological space X coincides with the the jth homology of the **algebraic** *deRham complex $\Omega^\bullet_{A/\mathbf{C}}$ for $A = \mathbf{C}[T_1, \dots, T_n]/(f_1, \dots, f_m)$. That is, all degree-$j$ cohomology classes on X are represented by elements of $\ker(\Omega^j_{A/\mathbf{C}} \to \Omega^{j+1}_{A/\mathbf{C}})$ and the cohomology class of a closed global algebraic j-form ω vanishes if and only if ω is "algebraically exact" in the sense that $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^{j-1}_{A/\mathbf{C}}$.*