Math 210B. Induced representations and A_5 character table

1. Induced representations

Let G be a finite group, H a subgroup, and (W,ρ) a finite-dimensional representation of H over a field $k=\overline{k}$. (Our main case of interest is when $\mathrm{char}(k)=0$, but for now it isn't necessary.) In class we discussed two ways to define/describe the *induced representation* $\mathrm{Ind}_H^G(\rho)$ of G on a vector space of dimension $[G:H]\dim W$. The algebraic approach is

$$\mathrm{Ind}_H^G(\rho)=k[G]\otimes_{k[H]}W$$

using the left $k[G]$-multiplication on itself (and for the tensor product in the sense over non-commutative but associative rings in Exercise 3(i) of HW9 we use the right-multiplication of $k[H]$ on $k[G]$ and the left-action of $k[H]$ on W via ρ). The function-theoretic approach is

$$\mathrm{Ind}_H^G(\rho)=\{f:G\to W\mid f(hx)=\rho(h)(f(x))\text{ for all }x\in G,h\in H\}$$

on which g acts via the recipe $(g.f)(x)=f(xg)$. In Exercise 3(ii) of HW9 you show that these two constructions are G-equivariantly isomorphic via $f\mapsto\sum_{\overline{g}\in G/H}[g]\otimes f(g^{-1})$.

Example 1.1. If ρ is the trivial action, then we get $\mathrm{Ind}_H^G(\mathbf{1}_H)=\{f:H\backslash G\to k\}$ on which G acts by $(g.f)(x)=f(xg)$. For the space of cosets G/H, we we have natural isomorphism of G-representation spaces

$$\mathrm{Map}(H\backslash G,k)=k[G/H]$$

assigning to such f the element $\sum_{\overline{g}\in G/H}f(g^{-1})[\overline{g}]$. For example, if $H=\{1\}$ then we see that $\mathrm{Ind}_H^G(\mathbf{1}_H)=k[G]$ is the regular representation, a rather extreme case of an irreducible representation of a subgroup having an extremely reducible induction. In general, as noted in class, $k[G/H]$ of dimension $[G:H]$ is always reducible when $H\neq G$ since the line of constant functions (i.e., the k-span of $\sum_{\overline{g}\in G/H}[\overline{g}]$) is a nonzero proper subrepresentation.

Pushing the end of the preceding example further, we now explore some necessary conditions for $\mathrm{Ind}_H^G(\rho)$ to be irreducible for G when ρ is irreducible for H. (Mackey theory will provide a more systematic approach to this issue.) The basic principle is that if ρ extends to an action of a strictly larger subgroup of G on the same space W then the induction all the way up to G is *never* irreducible, ultimately for a reason similar to the "constant functions" feature of the preceding example. Hence, if we want to hope that $\mathrm{Ind}_H^G(\rho)$ may be irreducible, it is a good idea to first explore if the action may extend to that of any intermediate groups we happen to know about.

To see what the issue is, consider the "extreme" case that $\rho:H\to\mathrm{GL}(W)$ happens to extend to an action $\widetilde{\rho}:G\to\mathrm{GL}(W)$ of the entirety of G on the same space W. We'll use this $\widetilde{\rho}$ to give an alternative description of $\mathrm{Ind}_H^G(\rho)$ in terms of which reducibility will always be visible (assuming $H\neq G$, of course). As motivation, for any function $f:G\to W$ belonging to the function space $\mathrm{Ind}_H^G(\rho)$, we can define a new function $\widetilde{f}:G\to W$ via the formula

$$\widetilde{f}(g)=\widetilde{\rho}(g)^{-1}(f(g))\in W.$$

This latter function is left H-invariant precisely because of the transformation law on f and the hypothesis $\widetilde{\rho}|_H=\rho$:

$$\widetilde{f}(hg)=\widetilde{\rho}(hg)^{-1}((f(hg))=\widetilde{\rho}(g)^{-1}(\widetilde{\rho}(h)^{-1}(\rho(h)(f(g))))=\widetilde{\rho}(g)^{-1}(f(g))=\widetilde{f}(g).$$

Thus, $\widetilde{f}$ naturally belongs to the space $\mathrm{Map}(H\backslash G,W)$ of W-valued functions on the coset space $H\backslash G$ (whose members are cosets Hg for $g\in G$). Since we can reconstruct f from $\widetilde{f}$ via the formula $f(g)=\widetilde{\rho}(g)(\widetilde{f}(g))$ due to how $\widetilde{f}$ was defined, we are led to discover:

Proposition 1.2. *Suppose there exists a representation $\widetilde{\rho} : G \to \mathrm{GL}(W)$ of G on W whose H-restriction coincides with ρ. For any $\varphi \in \mathrm{Map}(H\backslash G, W)$, the function $\varphi' : G \to W$ defined by $g \mapsto \widetilde{\rho}(g)(\varphi(Hg))$ belongs to $\mathrm{Ind}_H^G(\rho)$ and the resulting map of k-vector spaces*

$$\theta : \mathrm{Map}(H\backslash G, W) \to \mathrm{Ind}_H^G(\rho)$$

defined by $\varphi \mapsto \varphi'$ is a G-equivariant isomorphism, where the G-action on the source is defined by $(g.\varphi)(x) = \widetilde{\rho}(g)(\varphi(xg))$ for $x \in H\backslash G$.

That θ is a G-equivariant map is a matter of unraveling definitions, which we leave to the reader to check. The surjectivity we seen above (namely $\theta(\widetilde{f}) = f$), so θ is an isomorphism since both sides have the same dimension (namely $[G : H] \dim W$). We emphasize that the description of $\mathrm{Ind}_H^G(\rho)$ in this result depends *crucially* on being handed $\widetilde{\rho}$ extending the given H-action on W to a G-action on W; typically for a given representation (W, ρ) of H no such $\widetilde{\rho}$ exists. (In the special case $W = \mathbf{1}_H$ we can take $\widetilde{\rho}$ to be $\mathbf{1}_G$, so we obtain the G-equivariant description of $\mathrm{Ind}_H^G(\mathbf{1}_H)$ as $\mathrm{Map}(H\backslash G, k) \simeq k[G/H]$ discussed above.)

Inside the G-representation space $\mathrm{Map}(H\backslash G, W)$ is the subspace of *constant maps*: functions $H\backslash G \to W$ carrying all cosets to the same $w \in W$. This is clearly a G-stable subspace, and its dimension is $\dim W$ (the only "parameter" is the choice of $w \in W$). But the entire space $\mathrm{Map}(H\backslash G, W)$ has dimension $[G : H] \dim W$, and this is larger than $\dim W > 0$ whenever $H \neq G$. Thus, in this way we see that $\mathrm{Ind}_H^G(\rho)$ is *reducible* whenever there exists $\widetilde{\rho}$ extending the H-action on W to a G-action and $H \neq G$. This conclusion can be applied more widely, via the transitivity of induction:

Corollary 1.3. *If the H-action on W extends to an action on W by some subgroup K of G strictly containing H then $\mathrm{Ind}_H^G(\rho)$ is reducible.*

Proof. We saw in class that for any subgroup G' of G and any G'-representation V', $k[G] \otimes_{k[G']} V'$ is described concretely as a direct sum of copies of V' indexed by coset representatives for G/G'. In this way, one sees (why?) that $\mathrm{Ind}_{G'}^G$ is an exact functor. In particular, if $V_1' \subset V_2'$ is a proper inclusion of G'-representations then $\mathrm{Ind}_{G'}^G(V_1') \to \mathrm{Ind}_{G'}^G(V_2')$ is a proper inclusion (as is also seen via the function-theoretic description: it is certainly an injection, and the dimensions are not equal). In particular, the induction of a reducible representation is always reducible.

Now we apply the preceding to the inclusion of K into G. More specifically, since $\mathrm{Ind}_H^G(W) \simeq \mathrm{Ind}_K^G(\mathrm{Ind}_H^K(W))$, to prove reducibility of $\mathrm{Ind}_H^G(W)$ as a G-representation it suffices to prove reducibility of $\mathrm{Ind}_H^K(W)$ as a K-representation. By hypothesis H is a proper subgroup of K and the H-action on W extends to an action of K on W! Thus, by Proposition 1.2 and the discussion following it we have an alternative description of $\mathrm{Ind}_H^K(W)$ in terms of maps $H\backslash K \to W$, with the space of constant maps a nonzero proper H-subrepresentation of $\mathrm{Ind}_H^K(W)$. ∎

2. Characters of A_5

Now assume $\mathrm{char}(k) = 0$. We determine the character table of A_5; the induction of a *non-trivial* 1-dimensional character of A_4 will provide a crucial entry in the table. The 5 conjugacy classes in A_5 along with their respective sizes were listed in class: these are the respective classes of 1, (123), (12)(34), (12345), and (12354) with respective sizes 1, 20, 15, 12, and 12. We noted that since A_5 is simple, its only 1-dimensional representation is the trivial one, so all others have dimension at least 2. There are 4 more irreducible representations to be found (since there are 5 conjugacy classes), and their respective dimensions are 3, 3, 4, 5 because of the fortuitous accident that these four positive integers are the only solution (up to rearrangement) of the equation

$$60 = \#A_5 = 1 + n_2^2 + n_3^2 + n_4^2 + n_5^2$$

(even without using the condition $n_j \geq 2$, but one may as well use this known property to save time on the case-checking of possibilities). We denote these irreducible representations $\rho_3, \rho_3', \rho_4, \rho_5$ (indexing by the dimension).

The representation ρ_4 can be made rather directly: S_n acts on $\mathbf{C}^n$ via permutation of the standard basis, so it preserves the hyperplane $H = \{\sum x_i = 0\}$ with complement $\mathbf{C}(1, 1, \ldots, 1)$ on which S_n acts trivially. Hence, the character χ of the S_n-action on H satisfies

$$1 + \chi(g) = \#\mathrm{Fix}(g)$$

where $\mathrm{Fix}(g)$ is the number of fixed points of g on $\{1, 2, \ldots, n\}$. This allows one to compute χ for S_5 acting on the 4-dimensional $H \subset \mathbf{C}^5$, and hence for A_5 acting on H, from which one can verify without much pain (using the conjugacy classes of A_5 and their sizes) that $\langle \chi, \chi \rangle_{A_5} = 1$.

Remark 2.1. Here is a more enlightening way to see that the hyperplane $\{\sum x_i = 0\} \subset \mathbf{C}^5$ must be irreducible for A_5. The only 1-dimensional representation of A_5 is the trivial one, and all other irreducible representations have dimension at least 3 (!), but 3 is so close to $4 = \dim(H)$ that if H were reducible for A_5 then it would have to contain a 1-dimensional A_5-subreprepresentation, which is to say a line on which A_5 acts trivially. Thus, to rule this out it suffices to show the ambient $\mathbf{C}^5$ has its diagonal line (complementary to H) as the *only* line with trivial A_5-action.

The space $(\mathbf{C}^5)^{A_5}$ of A_5-invariant vectors is S_5-*stable* since A_5 is normal in S_5, and as such it is a representation space for $S_5/A_5 = \{\pm 1\}$, so it is a direct sum of isotypic parts for the only two irreducible representations of that order-2 quotient. Thus, $(\mathbf{C}^5)^{A_5}$ is a direct sum of its S_5-invariants and the space of vectors on which S_5 acts through the sign character. But obviously the only line in $\mathbf{C}^n$ with *trivial* S_n-action is the diagonal line, so it remains just to rule out the sign character of S_5 from occurring in $\mathbf{C}^5$. Consider a nonzero vector $v = (a_1, \ldots, a_5) \in \mathbf{C}^5$ with the property $(a_{g(i)}) = \mathrm{sgn}(g)(a_i)$ for all $g \in S_5$. This says that swapping any two coordinates of v negates the vector. This forces $a_i = -a_j$ for all $i \neq j$, which is easily seen to be impossible if the a_i's are not all zero (impossible in any dimension ≥ 3).

So we now have two rows of the character table: the trivial character and $\chi_4 := \chi_{\rho_4}$ satisfying

$$1 + \chi_4(g) = \#\mathrm{Fix}(g)$$

for all $g \in A_5$. By inspecting the cycle decomposition in S_5, we thereby read off the character values for χ_4:

$$\chi_4(1) = 4, \chi_4((123)) = 1, \chi_4((12)(34)) = 0, \chi_4((12345)) = -1, \chi_4((12354)) = -1.$$

It remains to determine the 3-dimensional irreducible characters $\chi_3 = \chi_{\rho_3}$ and $\chi_3' = \chi_{\rho_3'}$ and the 5-dimensional $\chi_5 = \chi_{\rho_5}$.

For the 5-dimensional case, we make a concrete construction: observe that A_4 is an index-5 subgroup of A_5, so for any 1-dimensional character ψ of A_4 the induction $\mathrm{Ind}_{A_4}^{A_5}(\psi)$ is 5-dimensional. From our determination of the characters of A_4, there are 2 *non-trivial* choices of ψ (denoted $\varepsilon, \varepsilon^2 = \varepsilon^{-1}$ in class). We claim by pure thought that their inductions *must* be irreducible! Indeed, since the only smaller dimensions of irreducible representations of A_5 are 1, 3, and 4, and in particular no 2-dimensional ones, if a 5-dimensional representation of A_5 is reducible then by inspection of possibilities of dimensions it *must* contain a 1-dimensional irreducible, which is to say a copy of the trivial representation (the only 1-dimensional representation of A_5)! But by Frobenius reciprocity, the multiplicity of the trivial representation of A_5 in $\mathrm{Ind}_{A_4}^{A_5}(\psi)$ is the same as that of ψ in the trivial representation of A_4, so this multiplicity is zero whenever $\psi \neq 1$. That rules out reducibility when $\psi \neq 1$, so $\mathrm{Ind}_{A_4}^{A_5}(\varepsilon)$ is the (unique up to isomorphism) irreducible 5-dimensional representation of A_5. (By uniqueness, $\mathrm{Ind}_{A_4}^{A_5}(\varepsilon^{-1})$ is also this same representation, up to isomorphism.) On Homework 10, you'll compute the character χ_5 of this induction:

$$\chi_5(1) = 5, \chi_5((123)) = -1, \chi_5((12)(34)) = 1, \chi_5((12345)) = 0, \chi_5((12354)) = 0.$$

What about χ_3 and χ_3'? Let's denote the character values as follows:

$$\chi_3(1) = 3, \chi_3((123)) = x, \chi_3((12)(34)) = y, \chi_3((12345)) = z, \chi_3((12354)) = w$$

$$\chi_3'(1) = 3, \chi_3'((123)) = x', \chi_3'((12)(34)) = y', \chi_3'((12345)) = z', \chi_3'((12354)) = w'.$$

In particular, as we move down the column for each conjugacy class, we get the values x, x' in the column for (123), the values y, y' in the column for (12)(34), etc. The column orthogonality against the first column (i.e., the column for the conjugacy class of $1 \in A_5$) gives relations between x and x', between y and y', etc. More specifically:

$$x' = -x, y' = -2 - y, z' = 1 - z, w' = 1 - w.$$

Now switch to row orthogonality. The orthogonality of the rows for χ_3 and 1 says

$$3 + 20x + 15y + 12z + 12w = 0$$

and the orthogonality of the rows for χ_3 and χ_4 says

$$12 + 20x - 12z - 12w = 0.$$

Adding these gives the relation $15 + 40x + 15y = 0$, and the orthogonality of the rows for χ_3 and χ_5 gives the further relation $15 - 20x + 15y = 0$. These last two relations yield that $x = 0, y = -1$ and hence $z + w = 1$. It remains to find z and w.

Now comes a pleasant miracle by inspection of representatives of conjugacy classes: in A_5, each conjugacy class is its own inverse (i.e., g and g^{-1} are conjugate in A_5 for all $g \in A_5$; the analogue for S_n is clear via cycle types, but it is generally not true in A_n's). It follows $\chi(g^{-1})$ and $\chi(g)$ coincide for all characters χ of A_5. (In terms of working over $\mathbf{C}$, this says that the character values are in $\mathbf{R}$.) This simplifies the task of working out the equations expressing row orthogonality. In particular, since $w = 1 - z$, the orthogonality of χ_3 and χ_3' says exactly $z^2 - z - 1 = 0$. Thus,

$$z = (1 \pm \sqrt{5})/2, \;\; w = 1 - z = (1 \mp \sqrt{5})/2.$$

The two choices of sign give rise to the rows for χ_3, χ_3' (it is a matter of convention which is labeled χ_3 and which is labeled χ_3': algebraically there is no way to distinguish the two square roots of 5), so this completes the determination of the character table for A_5.

Remark 2.2. If a group H is a *normal* subgroup of a group G, then for any $g \in G$ the effect of precomposition with g-conjugation c_g on H shuffles around the isomorphism classes of irreducible H-representations: $\rho \mapsto \rho \circ c_g$. On isomorphism classes this only depends on $gH \in G/H$ since conjugation on H by any $h_0 \in H$ has *no effect* on such isomorphism classes. Indeed, if $\rho : H \to \mathrm{GL}(V)$ is a representation then conjugating it against $\rho(h_0) \in \mathrm{GL}(V)$ gives a representation that is isomorphic to ρ and yet also literally equal to $\rho \circ c_{h_0}$ (so $\rho \simeq \rho \circ c_{h_0}$).

Applying this to $A_5 \subset S_5$, we get a natural action of S_5/A_5 on the set of isomorphism classes of irreducible representations of A_5. It is easy to check that conjugation against a transposition of S_5 swaps the conjugacy classes of (12345) and (12354), and in terms of χ_3 and χ_3' it follows that this resulting "outer automorphism" of A_5 swaps ρ_3 and ρ_3' (up to isomorphism, as representations of A_5). That provides an alternative explanation for why one cannot expect to algebraically distinguish ρ_3 and ρ_3'.