Let $\varphi : A \to B$ be a homomorphism between commutative rings. We have defined the *integral closure* A' of A in B to be the set of elements $b \in B$ satisfying $f(b) = 0$ for a *monic* $f \in A[T]$, and it was shown that A' is a subring of B. The formation of this integral closure is not so well-behaved under some operations on rings (such as formation tensor products against another A-algebra), but relative to localization it behaves well:

Proposition 0.1. *If $S \subset A$ is a multiplicative set then relative to the map of rings $S^{-1}A \to S^{-1}B$ the integral closure of $S^{-1}A$ in $S^{-1}B$ is the subring $S^{-1}A' \subset S^{-1}B$.*

Proof. That elements of the subring $S^{-1}A' \subset S^{-1}B$ are integral over $S^{-1}A$ is a simple calculation, as follows. Any element of $S^{-1}A'$ is of the form a'/s for some $s \in S$ and $a' \in A'$, so there exists $f = T^n + a_{n-1}T^{n-1} + \cdots + a_0 \in A[T]$ such that $f(a') = 0$ in B. That is,

$$a'^n + a_{n-1}a'^{n-1} + \cdots + a_0 = 0$$

in B. Now dividing through by s^n in $S^{-1}B$, we get

$$(a'/s)^n + (a_{n-1}/s)(a'/s)^{n-1} + (a_{n-2}/s^2)(a'/s)^{n-2} + \cdots + a_0/s^n = 0$$

in $S^{-1}B$. This is a monic relation for a'/s over $S^{-1}A$, namely a'/s is a root of of the monic polynomial

$$T^n + (a_{n-1}/s)T^{n-1} + (a_{n-2}/s^2)T^{n-2} + \cdots + a_0/s^n \in (S^{-1}A)[T]$$

over $S^{-1}A$.

The more interesting direction is to show that *any* element of $S^{-1}B$ integral over $S^{-1}A$ can be expressed as a fraction a'/s for some $a' \in A'$ and $s \in S$. That is, we suppose for some $b \in B$ and $t \in S$ that the fraction $b/t \in S^{-1}B$ is integral over $S^{-1}A$ and we seek to show $b/t = a'/t'$ for some such $a' \in A'$ and $t' \in S$.

The integrality hypothesis on b/t says that $h(b/t) = 0$ for some monic $h \in (S^{-1}A)[X]$, which is to say

$$h(X) = X^n + (a_{n-1}/s_{n-1})X^{n-1} + \cdots + (a_0/s_0)$$

for some $a_j \in A$ and $s_j \in S$. We can rewrite all fractions a_j/s_j and b/t with a common denominator in S (such as the product of t and all s_j's). That is, we can assume there is an $s \in S$ such that the given fraction in $S^{-1}B$ integral over $S^{-1}A$ is written as b/s for some $s \in S$ and $b \in B$ such that

$$(b/s)^n + (a_{n-1}/s)(b/s)^{n-1} + (a_{n-2}/s)(b/s)^{n-2} + \cdots + a_0/s = 0$$

in $S^{-1}B$ for some $a_0, \ldots, a_{n-1} \in A$. Multiplying through by s^n, this gives

$$b^n + a_{n-1}b^{n-1} + (sa_{n-2})b^{n-2} + \cdots + (s^{n-1}a_0) = 0$$

in $S^{-1}B$. But the left side is an element of B, so its vanishing in $S^{-1}B$ means that it is killed in B by multiplication against some $s' \in S$. Multiplying through by s'^n then kills it in B just as well, giving the vanishing property

$$(s'b)^n + (s'a_{n-1})(s'b)^{n-1} + (s'^2 sa_{n-2})(s'b)^{n-2} + \cdots + (s'^n s^{n-1}a_0) = 0$$

in B (not just in $S^{-1}B$!). This expresses that the element $s'b \in B$ satisfies a monic polynomial relation over A, so $s'b \in A'$! Since $b/s = (s'b)/(s's) \in S^{-1}A'$, we are done. ∎

Now let A be a domain, with fraction field F and integral closure $\widetilde{A} \subset F$. For any multiplicative set $S \subset A - \{0\}$, the integral closure of $S^{-1}A$ in $S^{-1}F = F$ is equal to $S^{-1}\widetilde{A}$. In particular, if A is normal (i.e., $\widetilde{A} = A$) then $S^{-1}A$ is integrally closed for all such S. As a special case, if A is normal then so is $A_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of A. Here is a converse result working just at the maximal ideals:

Corollary 0.2. *If A is a domain with fraction field F and $A_{\mathfrak{m}}$ is normal for all maximal ideals $\mathfrak{m}$ of A then A is normal.*

Proof. We want to show that the inclusion $A \subset \widetilde{A}$ inside F is an equality. A map of A-modules is an isomorphism if and only if the induced map after localizing at each maximal ideal is an isomorphism, so it is equivalent to show that the localized inclusion $A_{\mathfrak{m}} \subset \widetilde{A}_{\mathfrak{m}}$ is an equality for all $\mathfrak{m}$. But by the Proposition, $\widetilde{A}_{\mathfrak{m}}$ is the normalization of $A_{\mathfrak{m}}$. By hypothesis this latter localization is normal, so it coincides with its own normalization. This gives the desired equality after localization at every maximal ideal, so the desired equality $A = \widetilde{A}$ holds as well. $\blacksquare$