Math 210B. Irreducible closed sets and prime ideals

Let J be a radical ideal in $k[X_1, \dots, X_n]$ for an algebraically closed field k, and let $Z = \underline{Z}(J) \subset k^n$ be the corresponding affine algebraic set. By the Nullstellensatz $\underline{I}(Z) = J$, and Z is non-empty if and only if $J \neq (1)$. Recall that in class we defined the Zariski topology on such Z via radical ideals in $A := k[X_1, \dots, X_n]/\underline{I}(Z)$ (with $Z = \mathrm{MaxSpec}(A)$) and saw that this coincides with the subspace topology from the Zariski topology on $k^n = \mathrm{MaxSpec}(k[X_1, \dots, X_n])$.

We also saw that the Zariski topology on k^n is very non-Hausdorff (with a base of open sets given by $U_f = \{f \neq 0\}$ for $f \in k[X_1, \dots, X_n]$, and $U_f \cap U_g = U_{fg} \neq \emptyset$ when $U_f, U_g \neq \emptyset$ since U_f is empty if and only if $f = 0$).

Example 0.1. In the special case $n = 1$, the Zariski topology on k^n is the *cofinite topology*: the open sets are the empty set and complements of finite sets. Indeed, this corresponds to the fact that for $f \in k[X]$ the zero locus of f in k is either finite (when $f \neq 0$) or the entire space (when $f = 0$), and that every finite subset $\{a_1, \dots, a_n\}$ in k arises in this way (using $f = \prod(X - a_i)$). This is rather vividly non-Hausdorff.

We defined a topological space Y to be *irreducible* if $Y \neq \emptyset$ and Y cannot be written as a union of two *proper* closed subsets, and we saw that this latter condition is equivalent to saying that all non-empty open subsets of Y are *dense*. The first interesting way in which the "geometry" of the Zariski topology encodes interesting algebraic information about (radical) ideals is:

Proposition 0.2. *For radical J as above, $\underline{Z}(J)$ is irreducible if and only if J is a prime ideal.*

Proof. The condition that $\underline{Z}(J)$ is non-empty is exactly the condition that $J \neq (1)$, so we may and do assume that this condition holds. Hence, to prove "$\Rightarrow$" we just have to check that if $ab \in J$ for $a, b \in k[X_1, \dots, X_n]$ then $a \in J$ or $b \in J$. By our calculation that a finite union of affine algebraic sets is an affine algebraic set, we see that

$$\underline{Z}(J, a) \cup \underline{Z}(J, b) = \underline{Z}((J, a)(J, b)) = \underline{Z}(J^2, aJ, bJ, ab) = \underline{Z}(J)$$

since $\mathrm{rad}(J^2) = J$ and $ab \in J$. This exhibits the irreducible (!) $\underline{Z}(J)$ as a union of two closed subsets, so one of these closed subsets must coincide with $\underline{Z}(J)$. By relabeling if necessary, we may assume $\underline{Z}(J, a) = \underline{Z}(J)$. Applying $\underline{I}$, the Nullstellensatz gives that $\mathrm{rad}(J, a) = \mathrm{rad}(J) = J$. But $a \in \mathrm{rad}(J, a)$, so $a \in J$.

For the converse, assume J is prime (so $\underline{Z}(J)$ is non-empty) and suppose $\underline{Z}(J) = Z \cup Z'$ for closed subsets Z, Z' of $\underline{Z}(J)$. We have to show that one of these closed subsets coincides with $\underline{Z}(J)$. Assume the inclusions of Z, Z' in $\underline{Z}(J)$ are *proper*, so by the Nullstellensatz the reverse inclusions of $J = \underline{I}(\underline{Z}(J))$ in $I := \underline{I}(Z)$ and $I' := \underline{I}(Z')$ are proper. Hence, we can choose $a \in I - J$ and $a' \in I' - J$. Then aa' vanishes on $Z \cup Z' = \underline{Z}(J)$, so $aa' \in J$ by the Nullstellensatz. But J is prime and by design $a, a' \notin J$, so we have a contradiction. ■

Example 0.3. We have noted above that the affine algebraic sets in k^1 are just the finite sets and the entire space. In particular, the *irreducible* affine algebraic sets in k^1 are just points.

What are the irreducible closed sets in the affine plane k^2? By the above result, this amounts to asking for a description of the prime ideals in $k[X, Y]$. Some obvious prime ideals are (0), (f) for *irreducible* f in the UFD $k[X, Y]$, and the maximal ideals $(X - a, Y - b)$. Geometrically, these examples correspond respectively to k^2, $\underline{Z}(f)$ for irreducible $f \in k[X, Y]$ ("irreducible algebraic plane curves"), and points $(a, b) \in k^2$. Later we will see that this list is exhaustive.

Already in k^3 it is hopeless to explicitly describe all irreducible closed sets. Informally, the problem is that not all "irreducible curves" in k^3 are given by an intersection of two distinct "irreducible surfaces", and more seriously it is difficult to detect when an intersection $\underline{Z}(f, g) = \underline{Z}(f) \cap \underline{Z}(g)$ of two distinct "irreducible surfaces" is irreducible.