Math 210B. Noether normalization theorem

Let k be an algebraically closed field, and $Z \subseteq k^n$ an irreducible closed subset. We seek a linear surjection $L : k^n \to k^d$ such that in the resulting commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & k^n \\ & \searrow^{f} & \downarrow L \\ & & k^d \end{array}$$

the map f is *finite surjective*. That is, if L has components $\ell_1, \dots, \ell_d \in (k^n)^*$ (these ℓ_j are simply linear forms in the x_i's) then the induced k-algebra pullback map

$$L^* : k[T_1, \dots, T_d] \to k[Z]$$

defined by $T_j \mapsto \ell_j$ is a module-finite injection. (The injectivity of this k-algebra map corresponds to f having dense image, but the integrality arising from module-finiteness forces f to be *closed* and thus to be surjective whenever it has dense image.) In class we saw how to do this when Z is an irreducible hypersurface (i.e., $Z = \underline{Z}(f)$ for irreducible $f \in k[x_1, \dots, x_d]$) with $d = n - 1$, and the case $Z = k^n$ is trivial (take $d = n$, L the identity map). We shall now handle the general case by building on the hypersurface case and using induction on n via closedness properties of "finite morphisms".

Consider the prime ideal $P = \underline{I}(Z) \subset k[x_1, \dots, x_n]$. We may assume $P \neq 0$ (as otherwise we are in the trivial case $Z = k^n$), so there exists a nonzero $f \in P$. But $k[x_1, \dots, x_n]$ is a UFD and P is prime, so some irreducible factor of f also lies in P. That is, P contains an *irreducible* f. In geometric terms, this is just the assertion that Z lies in a proper hypersurface V, and by irreducibility of Z it follows that Z must lie in one of the irreducible components of V, which is necessarily an irreducible hypersurface (since irreducible elements of a UFD generate a prime ideal).

Now consider the situation $Z \hookrightarrow V \hookrightarrow k^n$, where $k[V] \to k[Z]$ is surjective (quotient by (f) mapping onto quotient by P). By the settled hypersurface case there is a linear surjection $L : k^n \to k^{n-1}$ such that $L|_V : V \to k^{n-1}$ is finite (i.e., $L^* : k[T_1, \dots, T_{n-1}] \to k[V]$ is a module-finite injection). Thus, the composite map

$$Z \hookrightarrow V \to k^{n-1}$$

is also finite (as a composition of module-finite ring maps is again module-finite) though probably not surjective. Nonetheless, we know that the *image* Z' of Z in k^{n-1} is a *closed* set due to the finiteness of the map on coordinate rings, and it inherits irreducibility from Z. In algebraic terms,

$$k[Z'] = \text{image}(k[T_1, \dots, T_{n-1}] \to k[Z]),$$

so the injective map $k[Z'] \to k[Z]$ is module finite since $L|_Z : Z \to k^{n-1}$ is "finite" (noted above). In other words, we have the diagram

$$\begin{array}{ccc} Z & \longrightarrow & k^n \\ \downarrow & & \downarrow L \\ Z' & \longrightarrow & k^{n-1} \end{array}$$

in which the left map is a finite surjection, the map along the right side is a linear surjection, and Z' is an irreducible closed set in k^{n-1}. (Maybe $Z' = k^{n-1}$, or Z' is a hypersurface in k^{n-1}, or something else; hard to tell.)

Ah, now by *induction on n* there is a linear surjection $L' : k^{n-1} \to k^d$ for some $d \geq 0$ such that $L'|_{Z'} : Z' \to k^d$ is a finite surjection (i.e., the k-algebra pullback map on coordinate rings is a module-finite injection of a d-variable polynomial ring into $k[Z']$). Thus, the composite map

$$Z \to Z' \to k^d$$

is a finite surjection induced by the surjective linear composite map $L' \circ L : k^n \to k^{n-1} \to k^d$.

Strictly speaking, to carry out the induction we need to first check the base case $n = 1$. (Perhaps by logic games with the empty set we can begin with $n = 0$, but I'd rather not think about something as fake as that.) In this case our argument can be given directly in a way that works over any field with a domain quotient A of $k[x]$: either A is $k[x]$ or it is $k[x]/(f)$ for an irreducible $f \in k[x]$. In the first case we take $d = 1$ and use the identity map on the affine line (or on $k[x]$), whereas in the second case we see by inspection that $A = k[x]/(f)$ is finite-dimensional over k, so we can take $d = 0$ and use the unique k-algebra inclusion $k \hookrightarrow A$. (It would be a mistake to "ignore" the possibility $d = 0$, corresponding to Z a closed point, since otherwise the base of the induction will become slightly more complicated when k is not algebraically closed.)