Math 210B. Normal field extensions

1. A definition

In Exercise 7 of HW5 it was shown that for a finite extension of fields k'/k, the conditions that k'/k is the splitting field of a monic polynomial is equivalent to the condition that the images of all k-embeddings $k' \to \overline{k}$ coincide. This property is the definition of *normal* for such extensions. Here we wish to take up the generalization of this concept to algebraic extensions of fields k'/k not necessarily of finite degree.

There are (at least) three ways one might generalize normality to an algebraic extension k'/k:

(i) All k-embeddings $k' \to \overline{k}$ have the same image.
(ii) Every finite subextension of k'/k is contained in a finite subextension that is normal.
(iii) There is a set S of monic polynomials in $k[X]$ such that k'/k splits all $f \in S$ and every finite subextension of k'/k is contained in the splitting field of a product of finitely many elements of S.

Note that in (iii) we are forced to consider sets S rather than a single polynomial because for a finite set of monic polynomials $f_1, \dots, f_m$ we can split all f_j's by splitting the single polynomial $\prod_j f_j$ but no such trick is available to create infinite-degree extensions. Our aim here is to show that (i), (ii), and (iii) are equivalent; any algebraic k'/k satisfying such conditions is called *normal*. At the end we'll discuss the analogue of *normal closure*.

First we show (i) implies (ii). Since a compositum of *finite-degree* normal extensions is normal (as normality is equivalent to being a splitting field of a single monic polynomial in the finite-degree case, and splitting several polynomials simulatenously is the same as splitting their product), we just have to check that for all $a \in k'$, the field $k(a) \subset k'$ is contained in a finite-degree subextension of k'/k that is normal. Thus, if suffices to show that if $f \in k[X]$ is monic with a root ρ in k' then it splits in k'. For any root $r \in \overline{k}$, we can pick a k-isomorphism $k(\rho) \simeq k(r) \subset \overline{k}$. This makes $\overline{k}$ an algebraic extension of $k(\rho)$ (by algebraicity over k) that is algebraically closed, so it is an algebraic closure of $k(\rho)$. Thus, the algebraic extension $k'/k(\rho)$ embeds into $\overline{k}$ over this chosen inclusion $k(\rho) \hookrightarrow \overline{k}$. In this way we have made k-embeddings $j_r : k' \to \overline{k}$ whose images contain whatever root r of f in $\overline{k}$ we wish. But by (i) all k-embeddings $j : k' \to \overline{k}$ have the *same* image, so in particular if we pick one such j then $j(k')$ contains *all* such r. This implies that the polynomial $f \in k[X]$ splits over $j(k')$, so it splits over the extension k'/k as desired.

Next, we show (ii) implies (iii). For each $a \in k'$, by (ii) the subfield $k(a) \subset k'$ is contained in a normal subextension of k'/k of finite degree, which is to say $k(a) \subset F_a \subset k'$ where F_a/k is the splitting field of some monic polynomial $f_a \in k[X]$. The set S of such f_a's works in (iii).

Finally, we show (iii) implies (i). Let $K \subset \overline{k}$ be the subfield generated over k by the roots of all $f \in S$. By (iii), any k-embedding $j : k \to \overline{k}$ has image containing K. But in fact the containment $K \subset j(k')$ must be an equality. Indeed, by (iii) every element $a \in k'$ lies in a subfield $F \subset k'$ that is the splitting field of $\prod f_j$ for some $f_1, \dots, f_m \in S$, so $j(a)$ is contained in the splitting field of $\prod f_j$ inside $\overline{k}$. But this latter splitting field is the compositum of the splitting fields $F_i \subset \overline{k}$ of each f_j, yet $F_i \subset K$ for all i by design of K. Thus, $j(a) \in K$. Since a was arbitrary in k', we have $j(k') \subset K$ as desired. This concludes the proof of the equivalence of (i), (ii), and (iii).

2. Normal closure

If k'/k is a general algebraic extension, a *normal closure* of k'/k is an algebraic extension E/k' normal over k with the "minimality" property that any k-embedding $k' \to K$ into a normal extension K/k extends to an embedding $E \hookrightarrow K$. We claim such an E/k' exists and is unique up to (typically non-unique!) k'-isomorphism. (For k'/k of finite degree we make such E/k' as a splitting field; in general we need to do a bit more.)

Before showing existence, let's check uniqueness. Suppose E/k' and F/k' are normal closures of k'. But the minimality properties of each and the normality of each over k, there exist k'-embeddings $j : E \to F$ and $i : F \to E$. The composite map $i \circ j : E \to E$ is a k'-embedding that we claim is an isomorphism (so i is surjective and hence is an isomorphism, so the uniqueness up to isomorphism would be established). By normality of F/k, if $a \in E$ is an element and $f \in k[X]$ is its minimal polynomial then f splits in F. The injective $i : F \to E$ then carries the *finite* set of roots of f in F onto the finite set of such roots in E. Since a is one of those roots, it is hit by i. Thus, i is surjective as claimed.

Next, we construct a normal closure. In effect, it is a huge union of splitting fields. Pick a k-embedding $j_0 : k' \to \overline{k}$. For each $a \in k'$ we have the splitting field $F_a \subset \overline{k}$ of the minimal polynomial $m_a \in k[X]$ of a. Let E be the compositum inside $\overline{k}$ of the (huge) set of such extensions F_a of k. (That is, an element of E is a ratio of polynomial expressions over k in elements of finitely many F_a's.) We claim that E works. First we show E/k is normal by verifying condition (i) in the definition of normality.

Any k-embedding $\iota : E \to \overline{k}$ makes $\overline{k}$ an algebraic extension of E that is algebraically closed, so it makes $\overline{k}$ an algebraic closure of E. But the definition of E as a subfield of $\overline{k}$ does the same, so by uniqueness of algebraic closures there is an automorphism σ of $\overline{k}$ carrying the definition inclusion $E \hookrightarrow \overline{k}$ over to ι. Both of these inclusions of E into $\overline{k}$ are k-embeddings, so (!) σ must be a k-automorphism. As such, σ restricts to an automorphism of each splitting field $F_a \subset \overline{k}$ (of each $m_a \in k[X]$, for $a \in k'$), and hence to their compositum E. By design $\sigma(e) = \iota(e)$ for all $e \in E$, yet we just saw that $\sigma(E) = E$, so $\iota(E) = E$. This verifies (i) in the definition of normality, so E/k' is normal over k.

Finally, we check the minimality requirement. Suppose $i : k' \to F$ is a k-embedding into a normal extension of k. We seek a k'-embedding $E \to F$. Since F is algebraic over k' and j_0 in the construction of E realizes $\overline{k}$ as an algebraic closure of k', we can extend j_0 through i to an embedding $\tau : F \to \overline{k}$; in other words, the latter is a k'-embedding (where $\overline{k}$ is an extension of k' via j_0, and F is an extension of k' via i). The image $\tau(F)$ is a subfield of $\overline{k}$ containing $j_0(k')$ and normal over k. For any $a \in k'$ the irreducible monic $m_a \in k[X]$ has a root $\tau(i(a)) \in \tau(F)$, so $\tau(F)$ contains the splitting field over k of m_a inside $\overline{k}$. This splitting field is precisely F_a as in the construction of E, so $E \subset \tau(F)$ as subfields of $\overline{k}$ containing $j_0(k')$. Since τ is a k'-embedding (intertwining i and j_0), the composite map $E \hookrightarrow \tau(F) \simeq F$ whose second step is τ^{-1} is a k'-embedding. This is the desired k'-embedding of E into any extension F/k' that is normal over k.