Math 210B. Spec

1. Some classical motivation

Let A be a commutative ring. We have defined the Zariski topology on the set $\mathrm{Spec}(A)$ of primes ideals of A by declaring the closed subsets to be those of the form

$$V(I) = \{\mathfrak{p} \supseteq I\}.$$

This is reminiscent of the classical situation where we worked with the set

$$k^n = \mathrm{MaxSpec}(k[t_1, \dots, t_n]))$$

for an algebraically closed field k (the equality with MaxSpec being essentially the Nullstellensatz) and declared the closed sets to be exactly the "affine algebraic sets"

$$\underline{Z}(I) = \{c \in k^n \mid f(c) = 0 \text{ for all } f \in I\} = \mathrm{MaxSpec}(k[t_1, \dots, t_n]/I).$$

In this handout, we will explore some topological and functorial features of $\mathrm{Spec}(A)$ in general that are analogues or more easily visualized features in the classical setting. This suggests that one can study problems with rather general commutative rings by using a "geometric intuition" acquired from visualizations modeled on the classical case. Such a viewpoint is incredibly powerful, proving geometric insight even into purely number-theoretic problems such as the study of Diophantine equations (including an understanding of local and global obstructions to solutions to such equations).

We first wish to understand the functoriality of $\mathrm{Spec}\, A$ in A. As a warm-up, we consider the classical case:

Example 1.1. Let k be an algebraically closed field, and let $Z \subset k^n$ and $Z' \subset k^m$ be Zariski-closed subsets. Let $f : Z \to Z'$ be a polynomial map, and let $f^* : k[Z'] \to k[Z]$ be the associated k-algebra map (defined by $f^*(h) = h \circ f$ in terms of k-valued functions). Then under the identifications

$$Z = \mathrm{MaxSpec}(k[Z]), \;\; Z' = \mathrm{MaxSpec}(k[Z'])$$

provided by the Nullstellensatz (denoted as $z \mapsto \mathfrak{m}_z$ and $z' \mapsto \mathfrak{m}_{z'}$), we claim that f can be reconstructed from f^* as follows:

$$f(\mathfrak{m}) = (f^*)^{-1}(\mathfrak{m}).$$

That is, $\mathfrak{m}_{f(z)} = (f^*)^{-1}(\mathfrak{m}_z)$ for all $z \in Z$.

To prove this, we simply compute. An element $h \in k[Z']$ lies in $\mathfrak{m}_{f(z)}$ if and only if $h(f(z)) = 0$. But $h(f(z)) = (h \circ f)(z) = (f^*(h))(z)$, so this vanishes if and only if $f^*(h) \in \mathfrak{m}_z$, which is to say $h \in (f^*)^{-1}(\mathfrak{m}_z)$.

Now we turn things around for Spec by taking the recipe in the preceding example as the *definition* of functoriality for Spec. More specifically, for a general ring map $\varphi : A \to B$ and a prime ideal $\mathfrak{p}$ of B, the preimage $\varphi^{-1}(\mathfrak{p})$ is a prime ideal of A (since the quotient ring $A/\varphi^{-1}(\mathfrak{p})$ is a subring of the domain $B/\mathfrak{p}$ and hence is a domain).

Beware that MaxSpec is *not* similarly behaved for general rings, in contrast with finitely generated algebras over a field k (and k-algebra maps!). For example, preimage under the injective map $k[T] \to k(T)$ carries the maximal ideal (0) of $k(T)$ to the non-maximal prime ideal (0) of $k[T]$. Note that in this example $k(T)$ is *not* finitely generated as a k-algebra. For an arbitrary field k, not necessarily algebraically closed, MaxSpec is well-behaved on finitely generated k-algebras precisely because of the Nullstellensatz. To be precise, consider a k-algebra map $\varphi : A \to B$ between such k-algebras, and $\mathfrak{m}$ a maximal ideal of B. The k-algebra domain $A/\varphi^{-1}(\mathfrak{m})$ is a k-subalgebra of the *field* $B/\mathfrak{m}$ that is finite-dimensional over k (Nullstellensatz!), forcing $A/\varphi^{-1}(\mathfrak{m})$ to be finite-dimensional over k and hence also a field. Thus, $\varphi^{-1}(\mathfrak{m})$ is maximal as desired. We are now led to:

Proposition 1.2. *For a ring map $\varphi : A \to B$, the induced map of sets $X(\varphi) : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ defined by $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$ is continuous, and $A \rightsquigarrow \mathrm{Spec}\, A$ is a contravariant functor from the category of rings to the category of topological spaces.*

Proof. The contravariant functoriality is straightforward: if φ is the identity then $X(\varphi)$ is clearly the identity map, and if $\psi : B \to C$ is another ring map then $X(\varphi) \circ X(\psi)$ carries a prime ideal $\mathfrak{q}$ of C to $\varphi^{-1}(\psi^{-1}(\mathfrak{q}) = (\psi \circ \varphi)^{-1}(\mathfrak{q}) = X(\psi \circ \varphi)(\mathfrak{q})$. That is, $X(\varphi) \circ X(\psi) = X(\psi \circ \varphi)$.

For the continuity, it suffices to check that the preimage of a closed set is closed. This is a computation: we claim that $X(\varphi)^{-1}(V(I)) = V(\varphi(I)B)$ for any ideal I of B. That is, if $\mathfrak{p}$ is a prime ideal of A then we claim that $X(\varphi)(\mathfrak{p}) \in V(I)$ if and only if $\mathfrak{p}$ contains $\varphi(I)B$, or equivalently if and only if $\varphi(I) \subset \mathfrak{p}$. This latter containment says exactly that $I \subset \varphi^{-1}(\mathfrak{p})$, and that in turn is precisely the statement that $X(\varphi)(\mathfrak{p}) \in V(I)$. ∎

2. Topological interpretation of algebra

As applications of the functoriality of Spec, we can identify certain closed or open subsets of $\operatorname{Spec} A$ with the spectrum of certain quotients or localizations of A:

Proposition 2.1. *Let $I \subset A$ be an ideal, and $a \in A$ be an element. The continuous map* $\operatorname{Spec}(A/I) \to \operatorname{Spec}(A)$ *induced by the natural map $A \twoheadrightarrow A/I$ is a homeomorphism onto $V(I)$, and the continuous map* $\operatorname{Spec}(A_a) \to \operatorname{Spec}(A) = X$ *induced by the natural map $A \to A_a$ is a homeomorphism onto the open set $X_a = X - V((a))$.*

Proof. The prime ideals of A/I are precisely $\mathfrak{p}/I$ for primes $\mathfrak{p} \in V(I)$, and the preimage of $\mathfrak{p}/I$ under $A \twoheadrightarrow A/I$ is exactly $\mathfrak{p}$, so $\operatorname{Spec}(A/I)$ is certainly carried bijectively onto $V(I)$. To see that this continuous bijection is a homeomorphism, we compute with closed sets. A closed set in $\operatorname{Spec}(A/I)$ is the set of primes containing an ideal $\overline{J}$ of A/I, and $\overline{J} = J/I$ for an ideal J of A containing I. Thus, $V(J)$ makes sense as a closed subset of $V(I)$, and it is easy to see (check!) that the bijection $\operatorname{Spec}(A/I) \to V(I)$ identifies $V(\overline{J})$ with $V(J)$.

The prime ideals of A_a are precisely $\mathfrak{p}A_a$ for primes $\mathfrak{p}$ of A that do not contain a, and under the map $A \to A_a$ the preimage of $\mathfrak{p}A_a$ is exactly $\mathfrak{p}$ (why?), so $\operatorname{Spec}(A_a)$ is carried bijectively onto X_a. To see that this continuous bijection is a homemorphism, we again compute with closed sets. A closed set in $\operatorname{Spec}(A_a)$ is the set $V_{A_a}(J)$ of primes containing an ideal J of A_a, and $J = I \cdot A_a$ where $I \subset A$ is the "ideal of numerators" of J (i.e., I is the set of $a' \in A$ such that $a'/a^n \in J$ for some $n \geq 0$). We claim that the closed set $X_a \cap V(I)$ in X_a is identified with $V_{A_a}(I \cdot A_a)$ under the continuous bijection $\operatorname{Spec}(A_a) \to X_a$. This says exactly that for a prime $\mathfrak{p}$ of A not containing a, $\mathfrak{p}$ contains I if and only if the prime ideal $\mathfrak{p} \cdot A_a$ of A_a contains $I \cdot A_a$. The implication "$\Leftarrow$" is obvious, and for the converse we note that if $I \cdot A_a \subset \mathfrak{p} \cdot A_a$ then for every $a' \in I$ necessarily $a' = x/a^n$ in A_a with some $x \in \mathfrak{p}$ and some $n \geq 0$. But then $a^n a'$ and x coincide in A_a, so for some $m \geq 0$ the equality $a^{m+n}a' = a^m x$ holds in A, so $a^{m+n}a' \in \mathfrak{p}$. Since $a \notin \mathfrak{p}$ and $\mathfrak{p}$ is prime, it follows that $a' \in \mathfrak{p}$ as desired. ∎

We can also interpret the irreducible closed sets in terms of prime ideals, analogous to the dictionary between irreducible affine algebraic sets and prime ideals in the classical setting:

Proposition 2.2. *The irreducible closed sets in $X = \operatorname{Spec} A$ are exactly $V(\mathfrak{p})$ for prime ideals $\mathfrak{p}$ of A, and each irreducible closed set $Z = V(\mathfrak{p})$ contains $\mathfrak{p}$ as its unique dense point.*

We sometimes call the dense point $\{\mathfrak{p}\}$ the *generic point* of $V(\mathfrak{p})$. Loosely speaking, we can visualize $\operatorname{Spec}(k[t_1, \dots, t_n])$ for $k = \overline{k}$ as being obtained from $\operatorname{MaxSpec}(k[t_1, \dots, t_n]) = k^n$ by adding in a new point $\{\mathfrak{p}\}$ for every classical irreducible closed set $\underline{Z}(\mathfrak{p})$, with this new point having closure whose closed points are precisely the points of the classical irreducible closed set of interest. (For example, in k^3 an irreducible surface acquires its own new dense point as well as a new generic point on every irreducible curve in the surface.)

Proof. Every irreducible closed set has the form $V(I)$ for a unique radical ideal I of A, and prime ideals are certainly radical, so the first assertion is that for radical I the closed set $V(I)$ is irreducible if and only if I is prime. Since $V(I)$ is homeomorphic to $\operatorname{Spec}(A/I)$, we can express everything in terms of the quotient ring $A' = A/I$: if A' is a reduced ring (i.e., no nonzero nilpotents) then $\operatorname{Spec} A'$ is irreducible if and only if A' is a domain. To prove this equivalence, we will imitate some of the calculations used in the proof of its classical counterpart (that $\underline{Z}(J)$ is irreducible if and only if J is prime, where J is a radical ideal of $k[t_1, \dots, t_n]$).

Suppose first that $\mathrm{Spec}(A')$ is irreducible (so it is non-empty, and hence $A' \neq 0$). To show that A' is a domain, we consider $a, b \in A'$ such that $ab = 0$ and we want to show that $a = 0$ or $b = 0$ in A'. Every prime contains $0 = ab$ and hence contains either a or b. That is, $\mathrm{Spec}(A') = V((a)) \cup V((b))$. This expresses the irreducible $\mathrm{Spec}(A')$ as a union of two closed subsets, so one of these subsets must be the entire space. That is, either $V((a))$ or $V((b))$ coincides with $\mathrm{Spec}(A')$, which is to say that either a lies in every prime or b does. However, the intersection of *all* prime ideals is the nilradical (set of nilpotent elements), which in A' is (0) since A' is assumed to be reduced. Hence, $(a) = (0)$ or $(b) = (0)$, so we get the vanishing of a or b. Conversely, if A' is a domain (hence nonzero) then $\mathrm{Spec}(A')$ is certainly non-empty and so to show it is irreducible we just need to show that if a pair of closed sets Z, Z' cover the entire space then one of them is the entire space. In fact, in such cases there is the point $\eta = \{(0)\}$ (since A' is a domain!) that I claim is *dense*. Granting this, the dense point must lie in one of the closed sets Z or Z' that are assumed to cover the entire space, but then by density of this point we see that whichever of the *closed* sets Z or Z' contain this point must in fact be the entire space. It remains (for the characterization of irreducible closed sets in terms of prime ideals) to show that $\{(0)\}$ is dense in $\mathrm{Spec}\, A$ when A is a domain. In other words, we claim that the only closed set Z which contains $\{(0)\}$ is the entire space. Indeed, if $Z = V(I)$ contains $\{(0)\}$ then by definition $I \subseteq (0)$, so $I = (0)$ and hence $Z = V((0))$ is the whole space.

Now we turn to the other assertion: $\{\mathfrak{p}\}$ is the unique dense point in $V(\mathfrak{p})$. Since $V(\mathfrak{p})$ is closed in the entire space, this assertion is intrinsic to the topological space $V(\mathfrak{p})$. Under the identification of $V(\mathfrak{p})$ with $\mathrm{Spec}(A/\mathfrak{p})$ as topological spaces, we can rename $A/\mathfrak{p}$ as A and replace $\mathfrak{p}$ with (0) to reduce to the following claim: if A is a domain then $\{(0)\}$ is the unique dense point of $\mathrm{Spec}(A)$. We saw above that this point is dense, so it remains to show that it is the only dense point. That is, if $\mathfrak{p}$ is a *nonzero* prime of the domain A then we claim that the closure of $\{\mathfrak{p}\}$ is not the entire space. But we have seen that the closure is exactly $V(\mathfrak{p})$, so we just have to check that if $\mathfrak{p} \neq (0)$ then $V(\mathfrak{p}) \neq \mathrm{Spec}(A)$. But this is clear, as the point $\{(0)\}$ is certainly *not* in $V(\mathfrak{p})$ when $\mathfrak{p} \neq (0)$ (why?). ■