Math 210B. Homework 10

1. Compute the character of the unique irreducible 5-dimensional representation $\mathrm{Ind}_{A_4}^{A_5}(\varepsilon)$ of A_5, where we view A_4 as the A_5-stabilizer of $5 \in \{1,2,3,4,5\}$ and $\varepsilon : A_4 \to \mathbf{C}^\times$ is either of the two non-trivial 1-dimensional characters of A_4. (Hint: for $\sigma = (12345)$, show $\{\sigma^j\}_{j \in \mathbf{Z}/(5)}$ represents A_5/A_4 and $\sigma^{-j} g \sigma^j \in A_4 = \mathrm{Stab}_{A_5}(5)$ if and only if g fixes $\sigma^j(5) = j \bmod 5$.)

2. For a field F, let $H = H(F)$ be the group of upper-triangular unipotents in $\mathrm{GL}_3(F)$; this is often called the *Heisenberg group* (a special case of a general construction used by Weyl to relate the approaches to quantum mechanics by Schrödinger and Heisenberg).

(i) Show $\{h = (h_{ij}) \in H \mid h_{12}, h_{23} = 0\}$ is the center Z of H, and $H/Z \simeq V := F \oplus F$ via the matrix entries h_{12}, h_{23}. For a group G with center C such that $A := G/C$ is abelian, show the commutator $G \times G \to G$ factors through an alternating bi-additive $b_G : A \times A \to C$. Identifying Z with F via the matrix entry h_{13}, show $b_H = \det : V \times V \to F$.

(ii) Now assume F is finite. Letting $A \subset H$ be the abelian subgroup defined by $h_{12} = 0$, consider $\chi \in \widehat{A}$ such that $\chi|_Z \neq 1$. Defining $\chi^H := \mathrm{Ind}_A^H(\chi)$, for such $\chi_1, \chi_2 \in \widehat{A}$ use Mackey theory to show: $\langle \chi_1^H, \chi_2^H \rangle_H = 1$ when $\chi_1|_Z = \chi_2|_Z$, and $\langle \chi_1^H, \chi_2^H \rangle_H = 0$ otherwise.

(iii) In the setting of (ii), prove χ^H is irreducible with central character $\chi|_Z \neq 1$ and that $\chi_1^H \simeq \chi_2^H$ if and only if $\chi_1|_Z = \chi_2|_Z$. (By Schur's Lemma, for any irreducible representation ρ of a finite group G over $\mathbf{C}$, the center Z acts through ρ by scaling against a homomorphism $\omega : Z \to \mathbf{C}^\times$; we call ω the *central character* of ρ.)

(iv) Via sum-of-squares, show the χ^H's and 1-dimensional characters of H/Z account for all irreducible representations of H. Deduce that for *every* nontrivial $\omega \in \widehat{Z}$ there is a unique irreducible representation of H with central character ω (Stone–von Neumann Theorem).

3. (i) If (V, ρ) is a finite-dimensional representation of a finite group G over a field k of characteristic 0 and $\chi := \chi_V$ then prove (cf. Lemma 2.2 in the Frobenius–Schur handout):

$$\chi_{\wedge^3(V)}(g) = \frac{\chi(g)^3}{6} - \frac{\chi(g^2)\chi(g)}{2} + \frac{\chi(g^3)}{3}, \quad \chi_{\mathrm{Sym}^3(V)}(g) = \chi(g^2)\chi(g) + \chi_{\wedge^3(V)}(g).$$

[See Exercise 9.3(a) in §9.1 of Serre's book *Linear representations of finite groups* for exponential generating functions that generalize this to all higher symmetric and exterior powers.]

(ii) If (V, ρ) and (V', ρ') are irreducible representations (say over $\mathbf{C}$) of a finite group G, why is the multiplicity in $V \otimes V'$ of an irreducible representation (W, θ) equal to $\langle \chi_\rho \chi_{\rho'}, \chi_\theta \rangle_G$? Show that the multiplicity of θ in $\rho \otimes \rho'$ coincides with the multiplicity of ρ' in $\rho^* \otimes \theta$.

4. (i) Construct a natural isomorphism $\bigoplus_{a+b=n} \mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(V') \simeq \mathrm{Sym}^n(V \oplus V')$ for finite-dimensional vector spaces V, V' over a field k. Use these to define a natural k-algebra isomorphism $\mathrm{Sym}(V) \otimes \mathrm{Sym}(V') \simeq \mathrm{Sym}(V \oplus V')$ (where $\mathrm{Sym}(V) := \bigoplus_{n \geq 0} \mathrm{Sym}^n(V)$, etc.).

(ii) Let $\rho : G \hookrightarrow \mathrm{GL}(V) \subset \mathrm{End}(V) = V \otimes V^*$ be a *faithful* representation of a finite group. Show that this map of sets is equivariant for the left G-translation on G and the G-action on $W := V \otimes V^*$ through the action on the left tensor factor V (no effect on V^*!).

(iii) Assume $k = \overline{k}$, $\#G \in k^\times$, and view $G \hookrightarrow W$ in (ii) as a finite subset of an affine space $\underline{W}$ over k. Show the coordinate ring map $k[\underline{W}] = \mathrm{Sym}(W^*) \to \prod_{g \in G} k$ is surjective (hint: Nullstellensatz). Use a basis $\{\ell_1, \dots, \ell_d\}$ of V^* to identify this with a G-equivariant map $k[\underline{V}]^{\otimes d} \twoheadrightarrow k[G]$ onto the regular representation and deduce *every* irreducible representation of G is a subrepresentation of a tensor power of (V^*, ρ^*) (so of (V, ρ) by double duality).