

MATH 210B. HOMEWORK 11 (NOT TO BE TURNED IN!)

1. Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ an exact sequence of modules over a ring A , $I \subset A$ an ideal. Let M^\wedge be the I -adic completion of M (and likewise for M', M'').

(i) For a sequence $\{x_n\}_{n \geq 1}$ in M^\wedge , show $\sum_{n \geq 1} x_n$ converges in M^\wedge if and only if $x_n \rightarrow 0$.

(ii) Show $M^\wedge \rightarrow M''^\wedge$ is surjective with kernel the closure of $\text{image}(M' \rightarrow M^\wedge)$.

(iii) If the I -adic topology on M induces the I -adic topology on M' , show the closure in (ii) is the image of $M^\wedge \rightarrow M^\wedge$ and this map is injective. (Hint: Cauchy sequences.)

2. (i) For a ring A separated and complete for the topology of an ideal I , and $f \in A[x]$ and $a_0 \in A$ with $f(a_0) \equiv 0 \pmod{I}$ and $f'(a_0) \in (A/I)^\times$ ($a_0 \pmod{I}$ is a “simple root” of f in A/I), prove there is a unique $a \in A$ with $a \equiv a_0 \pmod{I}$ and $f(a) = 0$. (Hint: if $f(a_{n-1}) \equiv 0 \pmod{I^n}$ and $t_n \in I^n$, show $f(a_{n-1} + t_n) \equiv f(a_{n-1}) + f'(a_{n-1})t_n \pmod{I^{n+1}}$.) This is *Hensel’s Lemma*.

(ii) As an application of (i) show $1 + x$ is a square in $k[[x]]$ when $\text{char}(k) \neq 2$, and -1 is a square in \mathbf{Z}_5 . Compute $c_1, c_2 \in \{0, 1, 2, 3, 4\}$ such that $(2 + 5c_1 + 25c_2)^2 \equiv -1 \pmod{5^3}$.

(iii) Using (i), prove the natural map $\mathbf{Z}_p^\times \rightarrow \mathbf{F}_p^\times$ has a unique *multiplicative* section, so for $c \in \mathbf{F}_p^\times$ there is a unique $[c] \in \mathbf{Z}_p$ so that $[c] \pmod{p} = c$ and $[c]^{p-1} = 1$. Compute $[2] \in \mathbf{Z}_7^\times \pmod{7^3}$, and show each $x \in \mathbf{Z}_p$ has the *unique* form $\sum_{n \geq 0} [c_n]p^n$ with $c_n \in \mathbf{F}_p$ and $[0] := 0$.

3. Let (A, \mathfrak{m}) be local noetherian, B a module-finite A -algebra. For any maximal ideal $\mathfrak{n} \subset B$, prove $\mathfrak{m}B_{\mathfrak{n}}$ contains some $(\mathfrak{n}B_{\mathfrak{n}})^e$, so the $\mathfrak{m}B_{\mathfrak{n}}$ -adic and $\mathfrak{n}B_{\mathfrak{n}}$ -adic topologies on $B_{\mathfrak{n}}$ coincide. (Hint: $\dim_{A/\mathfrak{m}}(B/\mathfrak{m}B) < \infty$.) For the $\mathfrak{m}B$ -adic completion B^\wedge , prove $B^\wedge \rightarrow \prod_{\mathfrak{n}} (B_{\mathfrak{n}})^\wedge$ is an isomorphism by using the local decomposition for the artinian rings $B/\mathfrak{m}^n B$ (all $n \geq 1$).

4. Let $f \in k[x, y]$ be irreducible, with $k = \bar{k}$. We say that the irreducible plane curve $Z = \underline{Z}(f)$ is *smooth* at $\xi = (a, b) \in Z$ when at least one of $(\partial_x f)(\xi)$ or $(\partial_y f)(\xi)$ is nonzero, and then call the line $L_\xi = \{(\partial_x f)(\xi)(x - a) + (\partial_y f)(\xi)(y - b)\}$ through ξ the *tangent line* to Z at ξ . Assuming Z is smooth at all ξ , we’ll prove $k[Z]$ is *Dedekind*.

(i) Show that for any line ℓ in k^2 through ξ *distinct* from the tangent line, the maximal ideal \mathfrak{m}_ξ of the 1-dimensional noetherian local domain $\mathcal{O}_{Z, \xi}$ is generated by ℓ . (Hint: compute $\mathfrak{m}_\xi/\mathfrak{m}_\xi^2$ by changing coordinates so $\xi = (0, 0)$, $L_\xi = x$, and $\ell = y$.)

(ii) Show $k[T]/(T^n) \rightarrow \mathcal{O}_{Z, \xi}/\mathfrak{m}_\xi^n$ defined by $T \mapsto \ell$ is an isomorphism. Deduce $\mathcal{O}_{Z, \xi}^\wedge \simeq k[[T]]$ (a DVR!). For $Z = \{y^2 - x(x^2 + 1)\} \subset k^2$, $\xi = (0, 0)$, and $\ell = y$, make the isomorphism $\mathcal{O}_{Z, (0,0)}^\wedge \simeq k[[y]]$ explicit by computing x modulo y^6 (hint: feed “ $x = -x^3 + y^2$ ” into itself).

(iii) Via $A/I^n \simeq \widehat{A}/\widehat{I}^n$ for noetherian I -adic completions, prove $\mathcal{O}_{Z, \xi} - \{0\} = \cup_{n \geq 0} \mathcal{O}_{Z, \xi}^\times \cdot \ell^n$, so $\mathcal{O}_{Z, \xi}$ is a local PID. Conclude that $k[Z]$ is Dedekind when Z is smooth at all points.

(iv) For irreducible closed $Z \subset k^2$ with $k[Z]$ Dedekind, prove Z is smooth at all points.

5. Let A be a domain finitely generated over $k = \bar{k}$, $K = \text{Frac}(A)$, $m = \dim_K \Omega_{K/k}^1$.

(i) Show $\Omega_{A/k}^1$ is a finitely generated A -module. Using that $K \otimes_A \Omega_{A/k}^1 \simeq \Omega_{K/k}^1$, find $a \in A - \{0\}$ so that $\Omega_{A_a/k}^1 = (\Omega_{A/k}^1)_a$ is A_a -free with basis $\{df_1, \dots, df_m\}$ for some $f_j \in A_a$.

(ii) For $x \in \text{Max}(A)$ with $a(x) \neq 0$ and $t_j = f_j - f_j(x)$, show $k[[T_1, \dots, T_m]]/(T_1, \dots, T_m)^2 \rightarrow A/\mathfrak{m}_x^2$ via $T_j \mapsto t_j$ is *surjective* (hint: if a line in $\mathfrak{m}_x/\mathfrak{m}_x^2$ is not hit, find a *non-zero* k -derivation $A \rightarrow k$ killing all f_j). Deduce $\pi_x : k[[T_1, \dots, T_m]] \rightarrow A_{\mathfrak{m}_x}^\wedge$ is surjective. Krull proved $\dim A_{\mathfrak{m}_x}^\wedge = \dim A_{\mathfrak{m}_x}$ ($= \dim A$) and $\dim k[[T_1, \dots, T_m]]/J < m$ if $J \neq 0$, so all π_x are *isomorphisms* (“ A_a is k -smooth”) if K/k has a separating transcendence basis (i.e., $m = \dim A$).