Math 210B. Homework 2

1. (i) Read the "Norm and trace" handout, noting transitivity properties and Galois-theoretic formulas in the separable case (and how much harder transitivity is to prove for norm than for trace beyond the separable case). Using 3×3 matrices, show for $K = \mathbf{Q}(\alpha)$ with $\alpha^3 = 2$ that if $x = c_0 + c_1\alpha + c_2\alpha^2$ with $c_j \in \mathbf{Q}$ then $\mathrm{N}_{K/\mathbf{Q}}(x) = c_0^3 + 2c_1^3 + 4c_2^3 - 6c_0c_1c_2$.

(ii) If A is a nonzero commutative k-algebra with $\dim_k A < \infty$, define $\mathrm{Tr}_{A/k} : A \to k$ as for fields (i.e., $\mathrm{Tr}_{A/k}(a)$ is the trace of the k-linear $m_a : x \mapsto ax$). Show that if $A = k^n$ then $\mathrm{Tr}_{A/k}(c_1, \dots, c_n) = \sum c_j$, and that if k'/k is any extension field then $\mathrm{Tr}_{(A \otimes_k k')/k'} : A \otimes_k k' \to k'$ coincides with $\mathrm{Tr}_{A/k} \otimes \mathrm{id}_{k'}$. Applying this to $A = L$ a finite separable extension and k'/k an extension that splits L/k, deduce a *much slicker* proof of Theorem 2.5 in the handout.

2. For a field extension k'/k, show the isomorphism $k' \otimes_k (k[X_1, \dots, X_n]) \simeq k'[X_1, \dots, X_n]$ induces an injective map of rings $f : k' \otimes_k (k(X_1, \dots, X_n)) \to k'(X_1, \dots, X_n)$ via localization and that f is an isomorphism when k'/k is *algebraic* (hint: reduce to k'/k finite) but *never* surjective when k'/k is not algebraic.

3. (i) Show a finite extension K/k is separable if and only if $k' \otimes_k K$ is reduced for all *purely inseparable* finite k'/k (hint for "$\Leftarrow$": deduce the k_{p}-finite $k_{\mathrm{p}} \otimes_k K$ is reduced), and if K/k is *not* separable (allow $[K : k]_{\mathrm{s}} > 1$!) show $K \otimes_k K$ is non-reduced.

(ii) For $k = \mathbf{F}_p(s, t)$ (rational function field in 2 variables), prove $f = sX^p + tY^p - 1 \in k[X, Y]$ is irreducible and at least one of s or t is not a pth power in $k(a^{1/p})$ for any $a \in k - k^p$.

(iii) Let $A = k[X, Y]/(f)$ for k and f as in (ii), so $K = \mathrm{Frac}(A)$ is finitely generated over k. Show $\mathrm{trdeg}(K/k) = 1$ (with each of X and Y as a transcendence basis), $[K : k(X)] = p$, and $K \otimes_k k(a^{1/p})$ is a field (especially: reduced!) for any $a \in k - k^p$.

(iv) Continuing from (iii), show if $k' \subset K$ is non-trivial algebraic over k then k'/k is purely inseparable with $[k' : k] = p$. Using that $k' \otimes_k k'$ is non-reduced, get a contradiction and conclude that k is *algebraically closed in* K. Also show $k(s^{1/p}, t^{1/p}) \otimes_k K$ is non-reduced, so *K/k has no separating transcendence basis* (interesting since k algebraically closed in K!).

4. Let L_1 and L_2 be two finite extensions of k inside an extension field L/k.

(i) Construct a surjection $h : L_1 \otimes_k L_2 \to L_1L_2$. Deduce $[L_1L_2 : k] \le [L_1 : k][L_2 : k]$ with equality if and only if h is an isomorphism (we then say L_1, L_2 are *linearly disjoint* over k).

(ii) Prove that if $\gcd([L_1 : k], [L_2 : k]) = 1$ then equality holds in (i), and that the converse is false. For $k = \mathbf{Q}$ and $L = \mathbf{C}$, take L_1 and L_2 to be images of embeddings of $\mathbf{Q}[x]/(x^3 - 2)$ to show that equality can fail when $L_1 \cap L_2 = k$ (inside L) and that L_1L_2 and $L_1 \cap L_2$ are *not* generally determined up to k-isomorphism by the k-isomorphism classes of L_1 and L_2.

5. Let K/k be finite separable, and L/k an extension containing a Galois closure of K/k.

(i) Prove that the map $K \otimes_k L \to \prod_{j \in \mathrm{Hom}_k(K,L)} L$ defined by $a \otimes b \mapsto (j(a)b)$ is an *isomorphism* of L-algebras (Hint: primitive element theorem). We say L *splits* K over k.

(ii) Deduce in particular that if K/k is finite Galois with Galois group G then the natural map $K \otimes_k K \to \prod_{g \in G} K$ defined by $a \otimes b \mapsto (g(a)b)$ is an isomorphism of rings. If $k = \mathbf{Q}$ and $K = \mathbf{Q}(\sqrt{3})$, find the elements of $K \otimes_k K$ which correspond to $(1, 0)$ and $(0, 1)$ in the target $K \times K$. Verify by explicit computation that these elements are idempotent ($e^2 = e$).

(iii) In contrast with (ii), if $K' \subset \overline{k}$ is a subfield containing k and K' meets the Galois closure of K/k in exactly k, prove $K \otimes_k K'$ is a *field* (so K, K' are linearly disjoint over k).