

MATH 210B. HOMEWORK 3

1. If $K'/K/k$ are field extensions with K'/k finitely generated, prove K/k is finitely generated. Deduce that the algebraic closure of k in a finitely generated extension E (i.e., the set of $x \in E$ algebraic over k) is a *finite* extension. (Hint: induct on $d = \text{trdeg}(K'/k)$ to reduce to K/k algebraic, and if $\{x_1, \dots, x_d\}$ is a transcendence basis for K'/k then show $[K : k] \leq [K' : k(x_1, \dots, x_d)]$.) Nagata's counterexample to Hilbert's 14th Problem gives interesting k -subalgebras of $k[X_1, \dots, X_n]$ that are *not* finitely generated (as a k -algebra)!

2. Motivated by our study of finitely generated field extensions, a general field extension K/k is called *separable* if $k' \otimes_k K$ is reduced for all purely inseparable finite extensions k'/k .

(i) Assume $\text{char}(k) = p > 0$. Show separability is equivalent to each of the following: (a) $k_p \otimes_k K$ is reduced, (b) $k' \otimes_k K$ is reduced for all finite k'/k , (c) $\bar{k} \otimes_k K$ is reduced, (d) every *finitely generated* subextension over k admits a separating transcendence basis.

(ii) Assume $\text{char}(k) = p > 0$. Show $K := \bigcup k(X^{1/p^n}) \subset k(X)_p$ is separable but not finitely generated over k , and that each non-empty subset of K algebraically independent over k consists of exactly one element y yet the algebraic $K/k(y)$ is *never* separable.

(iii) For extensions $K/E/k$, show E/k is separable if K/k is (can fail for K/E ; see (ii)).

(iv) Show the formal Laurent series field $K = k((X))$ (i.e., fraction field of the formal power series ring $k[[X]]$) is separable over k by constructing a ring map $\varphi : k' \otimes_k (k[[X]]) \rightarrow k'[[X]]$ for any extension k'/k and proving φ is an isomorphism if $[k' : k] < \infty$ (it is false otherwise).

3. Let $A \rightarrow B$ be a ring map, and M a B -module. Define the A -algebra $B * M$ as in class.

(i) Prove that A -algebra sections to $\text{pr}_1 : B * M \rightarrow B$ are exactly $b \mapsto (b, D(b))$ for A -linear derivations $D : B \rightarrow M$.

(ii) For $b_1, \dots, b_n \in B$, $f \in A[X_1, \dots, X_n]$, and an A -linear derivation $D : B \rightarrow M$, show $D(f(b_1, \dots, b_n)) = \sum_j (\partial f / \partial X_j)(b_1, \dots, b_n) D(b_j)$. For $P_n := A[X_1, \dots, X_n]$, show A -linear derivations $D : P_n \rightarrow N$ to P_n -modules N are determined by $D(X_1), \dots, D(X_n) \in N$ and that $D(X_j)$'s can be arbitrary. Why does this imply $\Omega_{P_n/A}^1 = \bigoplus P_n dX_j$?

(iii) For multiplicative sets $S \subset B$, show any A -linear derivation $B \rightarrow M$ *uniquely extends* to an A -linear derivation $S^{-1}B \rightarrow S^{-1}M$. Deduce the natural $S^{-1}B$ -linear map $S^{-1}\Omega_{B/A}^1 \rightarrow \Omega_{S^{-1}B/A}^1$ is an isomorphism by constructing an inverse using universality of Ω^1 .

4. Let A be a ring, B and A' be A -algebras, $B' := A' \otimes_A B$, $J \subset B$ an ideal, $\bar{B} = B/J$.

(i) Show $\delta : B \rightarrow \Omega_{B'/A'}^1$ defined by $b \mapsto d(1 \otimes b)$ is an A -linear derivation, and that the resulting B' -linear map $f : A' \otimes_A \Omega_{B/A}^1 \rightarrow \Omega_{B'/A'}^1$ is an isomorphism. (Hint: show $(\Omega_{B'/A'}^1, \delta)$ and $(A' \otimes_A \Omega_{B/A}^1, b \mapsto 1 \otimes db)$ are initial for A -linear derivations $D : B \rightarrow N'$ into B' -modules.)

(ii) For any generating set $\{b_i\}$ of J , show the natural \bar{B} -linear map $\Omega_{B/A}^1 / J\Omega_{B/A}^1 \rightarrow \Omega_{\bar{B}/A}^1$ is surjective with kernel spanned over B by the db_i 's (Hint: mapping property of $(\Omega_{\bar{B}/A}^1, d)$.)

(iii) For a map of A -algebras $f : A' \rightarrow B$, prove $\Omega_{B/A}^1 \rightarrow \Omega_{B'/A'}^1$ is surjective with kernel spanned over B by the elements $d(f(a'))$ for $a' \in A'$. (Hint: mapping property of $(\Omega_{B'/A'}^1, d)$.)

5. For extensions $K/E/k$ with K/E finite *separable*, prove any k -linear derivation $D : E \rightarrow V$ to a K -vector space uniquely extends to a derivation $D' : K \rightarrow V$. (Hint: $K = E(a)$ and the separable minimal polynomial $f \in E[T]$ of a satisfies $D'(f(a)) = 0$.) Deduce that the natural K -linear $K \otimes_E \Omega_{E/k}^1 \rightarrow \Omega_{K/k}^1$ is an isomorphism. (Hint: mapping property of $(\Omega_{K/k}^1, d)$.)