Math 210B. Homework 4

1. (i) If X is a topological space and a subspace Y is irreducible, prove the closure $\overline{Y}$ is irreducible (e.g., $Y = \{y\}$; we will later see important examples with non-closed points y).

(ii) Show that any noetherian topological space X is quasi-compact (i.e., every open cover of X admits a finite subcover) and that any subspace $Y \subset X$ is noetherian.

(iii) Conversely to (ii), if every subspace of a topological space X is quasi-compact then prove X is noetherian.

2. Over a field $k = \overline{k}$ with $\text{char}(k) \neq 2$, decompose $\underline{Z}(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3) \subset k^2$ and $\underline{Z}(u^2 + v^2 - 1, u^2 - w^2 - 1) \subset k^3$ into irreducible components. Also show that $V := \{(t, t^2, t^3) \in k^3 \,|\, t \in k\}$ is closed in k^3 and compute $\underline{I}(V)$.

3. Let R be a ring, $I \subset R$ an ideal whose elements are nilpotent, and $k = \overline{k}$ a field.

(*i*) If $r \in R$ and $r \bmod I \in (R/I)^\times$, prove $r \in R^\times$. (Do not invoke the existence of maximal ideals in a non-zero ring.) Make this explicit for $R = k[X]/(X^3)$, $r = -1 + X$, $I = (X)$.

(*ii*) An element $e \in R$ is called *idempotent* if $e^2 = e$. Using e and $1-e$, show that specifying an idempotent is 'the same' as specifying an ordered decomposition $R \simeq R_1 \times R_2$ for rings R_1, R_2. Show that for every idempotent $\overline{e} \in R/I$, there is a unique idempotent $e \in R$ with $e \bmod I = \overline{e}$. (Hint: show $2\overline{e} - 1 \in (R/I)^\times$ and use (i).) Find all idempotents distinct from 0 and 1 in $R = k[X, Y]/(X(X-1)(X-\lambda))$ where $\lambda \in k - \{0, 1\}$, and determine the associated decomposition of R as a direct product in each case. Draw pictures.

(*iii*) If $Z \subset k^n$ is an affine algebraic set, prove every point has a connected neighborhood (so all connected components are open) and interpret (with proof!) idempotents in $k[Z] := k[X_1, \dots, X_n]/\underline{I}(Z)$ in terms of connected components of Z. Deduce that $k[Z]$ has only finitely many idempotents.

4. The proof of the Normal Basis Theorem for finite Galois K/k with infinite k in Ch. VI 13.1 has a gap at "Hence the determinants will not be 0 for all $x \in K \dots$". We fix this.

If $f \in k[x_1, \dots, x_n]$ vanishes on k^n (with $n > 0$), show $f = 0$. The Normal Basis Theorem says $K \simeq k[G]$ as left $k[G]$-modules, where $k[G]$ is the (associative) *group algebra*: $k[G] = \oplus_{g \in G} k[g]$ with $[g] \cdot [h] := [gh]$ for $g, h \in G$ and k central in $k[G]$. For left $k[G]$-modules V and W with finite k-dimension, identify $\text{Isom}_{k[G]}(V, W)$ with a locus "$f \neq 0$" in $\text{Hom}_{k[G]}(V, W)$. Applying $K \otimes_k (\cdot)$, use HW2 Exercise 5(ii) to conclude. Then treat the case of finite k.

5. This exercise applies Galois theory to commutative algebra. Let K/k be a finite Galois extension of fields, and V an arbitrary K-vector space equipped with a semi-linear action by $\Gamma = \text{Gal}(K/k)$ (i.e. $\gamma(v + v') = \gamma(v) + \gamma(v')$ and $\gamma(c \cdot v) = \gamma(c) \cdot \gamma(v)$ for $c \in K$).

(i) If $V = K \otimes_k V_0$ for a k-vector space V_0, prove existence and uniqueness of a semi-linear action satisfying $\gamma(c \otimes v_0) = \gamma(c) \otimes v_0$, and that the natural map $V_0 \to V^\Gamma$ is an isomorphism.

(ii) Show the natural map $\rho : K \otimes_k V^\Gamma \to V$ is an *isomorphism* by reducing to finite-dimensional V and using the Normal Basis Theorem (Exercise 4). This is called *Galois descent*. (Note that injectivity of ρ requires a proof. Hint for surjectivity: show the K-span of any Γ-orbit in V is a quotient of $\oplus K\mathbf{e}_\gamma$ on which Γ acts by $\gamma'(\sum c_\gamma \mathbf{e}_\gamma) := \sum \gamma'(c_\gamma)\mathbf{e}_{\gamma'\gamma}$.)

(iii) If k'/k is a separable algebraic extension and a k-algebra R is reduced, prove $k' \otimes_k R$ is reduced. (Hint: reduce to k'/k finite Galois, and apply Galois descent to the nilradical of $k' \otimes_k R$.) Deduce that if $J \subset R$ is any ideal then $k' \otimes_k \text{rad}(J) = \text{rad}(k' \otimes_k J)$. In particular, if k is *perfect* (e.g., characteristic 0 or finite) and R is reduced then $\overline{k} \otimes_k R$ is reduced!