Math 210B. Homework 5

1. For a field $k = \overline{k}$, radical ideals $I \subset k[\underline{x}]$ and $J \subset k[\underline{y}]$, and a polynomial map $f : \underline{Z}(J) \to \underline{Z}(I)$, show that $\ker(f^* : k[\underline{x}]/I \to k[\underline{y}]/J)$ is radical with zero locus equal to the Zariski closure of $f(\underline{Z}(J))$. Deduce that f^* is injective if and only if the image of f is *dense* for the Zariski topology on $\underline{Z}(I)$. Why is this geometrically reasonable?

2. Let k be a field, and $f \in k[x]$ monic of degree > 0. Let $h(x, y) = y^2 - f$ if $\mathrm{char}(k) \neq 2$ and $h(x, y) = y^2 - y - f$ if $\mathrm{char}(k) = 2$. Assume h viewed in $k(x)[y]$ has no root $y_0 \in k[x]$, and that f is squarefree if $\mathrm{char}(k) \neq 2$. Prove h is irreducible in $k(x)[y]$ and the integral closure of $k[x]$ in $K = k(x)[y]/(h)$ is $k[x, y]/(h)$.

3. Read the Spec handout and do the following for commutative rings A.

(i) Suppose A is finitely generated over a field k. If $k = \overline{k}$, explain why the formula $\mathrm{rad}(J) = \underline{I}(\underline{Z}(J))$ (where $\underline{Z}(J) := \{z \in \mathrm{Max}(A) \mid f(z) = 0 \text{ for all } f \in J\}$ via the natural inclusion $A \subset \mathrm{Func}(\mathrm{Max}(A), k)$) says $\mathrm{rad}(J) = \bigcap_{\mathfrak{m} \supset J} \mathfrak{m}$ for any ideal $J \subset A$. Then show the latter holds for *any* field k (hint: For $f \in \overline{A} := A/J$, show $\overline{A}_f$ is a finitely generated k-algebra. If f is not nilpotent, show $\overline{A}_f$ is nonzero and so admits a maximal ideal $\mathfrak{m}$, and then explain via the Nullstellensatz over k why the preimage of $\mathfrak{m}$ under $\overline{A} \to \overline{A}_f$ is maximal.).

(ii) For general A and any ideal $J \subset A$, prove $\mathrm{rad}(J) = \bigcap_{\mathfrak{p} \supset J} \mathfrak{p}$. (Hint: pass to A/J, and to show that f is nilpotent if f lies in all primes of A consider primes of A_f when $A_f \neq 0$.) Deduce $V(J) \subset V(J')$ inside $\mathrm{Spec}(A)$ if and only if $\mathrm{rad}(J') \subset \mathrm{rad}(J)$ inside A.

(iii) Assume A is noetherian. Prove that $\mathrm{Spec}(A)$ is noetherian, and deduce for ideals $J \subset A$ that every prime of A/J contains a minimal such prime, with the set of minimal primes of A/J a *finite* set.

(iv) Using Zorn's Lemma, show that every prime ideal of an arbitrary commutative ring contains a minimal prime ideal. Construct a (non-noetherian) ring A with infinitely many minimal prime ideals.

4. Let A be finitely generated over a general field k. Give $X_0 := \mathrm{MaxSpec}(A)$ the subspace topology from $X := \mathrm{Spec}(A)$, and for $Y \subset X$ let $Y_0 := Y \cap X_0$.

(i) Show that the closed sets in X_0 are precisely $\mathrm{MaxSpec}(A/J)$ for uniquely determined radical ideals $J \subset A$, and use the Nullstellensatz over k to prove $Z \mapsto Z_0$ is a bijection between the sets of closed subsets in X and X_0, with $Z \subset Z'$ if and only if $Z_0 \subset Z_0'$. Deduce the same bijectivity and inclusion relation assertions for open subsets (hint: show $(X - Z)_0 = X_0 - Z_0$). This tight relationship between Spec and MaxSpec is very specific to finitely generated algebras over fields (and for a wider class of rings called "Jacobson").

(ii) For closed $Z \subset X$, show Z is the closure of Z_0 and that Z is irreducible if and only if Z_0 with its subspace topology is irreducible. (For $k = \overline{k}$, this links spaces in classical algebraic geometry to schemes.)

5. Let A be a domain and F' an extension of its fraction field F. For any $a' \in F'$ that is integral over A, show that the *minimal polynomial* $m_{a'}$ of a' in $F[t]$ lies in $\widetilde{A}[t]$, where $\widetilde{A}$ is the integral closure of A in F. (Hint: if $f(a') = 0$ for a monic $f \in A[t]$, show the minimal polynomial of a' over F divides f in $F[t]$.)

Extra Credit: Conversely show if $a' \in F'$ is algebraic over F and $m_{a'} \in \widetilde{A}[t]$ then a' is integral over A.

6. Prove that any UFD is integrally closed, and for a field k prove that $k[X, Y]/(Y^2 - X^3)$ is a domain but not integrally closed. If $\mathrm{char}(k) \neq 2$, show that $R := k[x, y, z]/(xy - z^2)$ is an integrally closed domain but not a UFD (hint: $R = A[z]/(z^2 - a)$ for the UFD $A = k[x, y]$ and $a = xy$, so you can apply Exercise 5 with $F' = k(x, y)[z]/(z^2 - xy)$); extra credit for handling $\mathrm{char}(k) = 2$.

7. Let k'/k be a finite extension of fields, and $\overline{k}$ an algebraic closure of k.

(i) Prove that for any two k-embeddings $f_1, f_2 : k' \rightrightarrows \overline{k}$, there exists a k-automorphism $\theta : \overline{k} \simeq \overline{k}$ such that $\theta \circ f_1 = f_2$. (Hint: use uniqueness of algebraic closure up to isomorphism.)

(ii) Prove that k' is the splitting field of some monic polynomial $f \in k[T]$ if and only if the images of all k-embeddings $k' \to \overline{k}$ coincide inside $\overline{k}$; such k'/k are called *normal*. [Theorem 3.3 in Chapter V of Lang's *Algebra* uses a different viewpoint, resting on the crutch of a preferred k-embedding of k' into $\overline{k}$; it is better to avoid that, and not to look at Lang's proof.] The subfield of $\overline{k}$ generated over k by the images of the finitely many k-embeddings $f : k' \to \overline{k}$ is obviously normal, and is called the *normal closure* of k'/k.

(iii) If k'/k is normal then prove that its maximal separable subextension K/k is Galois (i.e., is the splitting field of a separable polynomial over k).