Math 210B. Homework 6

1. Let $K/\mathbf{Q}$ be an extension field of degree n, so $\mathscr{O}_K$ is finite free of rank n over $\mathbf{Z}$. A $\mathbf{Z}$-subalgebra $\mathscr{O} \subset \mathscr{O}_K$ is an *order* (in K) if it is finite free of rank n as a $\mathbf{Z}$-module.

(i) Prove that a subring $\mathscr{O}$ of $\mathscr{O}_K$ is an order in K if and only if it has finite index in $\mathscr{O}_K$, and that $\mathscr{O}$ always admits a $\mathbf{Z}$-basis containing 1.

(ii) For $\alpha \in \mathscr{O}_K$ prove that $\mathbf{Z}[\alpha]$ is an order in K if and only if $K = \mathbf{Q}(\alpha)$, in which case $\{1, \alpha, \dots, \alpha^{n-1}\}$ is a $\mathbf{Z}$-basis of this order.

(iii) Assume $n = 2$. For $f \geq 1$, prove that $\mathbf{Z} + f\mathscr{O}_K$ is the unique order of index f in $\mathscr{O}_K$. Give an explicit $\mathbf{Z}$-basis $\{1, \alpha_f\}$ of $\mathbf{Z} + f\mathscr{O}_K$ when $K = \mathbf{Q}(\sqrt{d})$ for a square-free $d \in \mathbf{Z}$.

2. Let k be a field, A an integrally closed domain finitely generated over k, and $F = \mathrm{Frac}(A)$.

(i) By HW3, Exercise 1, the algebraic closure of k in F is a finite extension of k; show it lies in A. Prove that if k is separably closed in F then for any separable algebraic extension k'/k, $F \otimes_k k'$ is a field in which k' is separably closed; in particular, $A \otimes_k k'$ is a domain (Hint: reduce to k'/k finite Galois, and use a primitive element and Galois descent.)

(ii) If k is separably closed in F, prove $A \otimes_k k'$ is integrally closed for k'/k separable algebraic. (Hint: reduce to k'/k finite, then Galois, and consider a $\mathrm{Gal}(k'/k)$-action on the integral closure.)

(iii) If k is imperfect with $p = \mathrm{char}(k) > 2$ and $a \in k - k^p$, show $A = k[x,y]/(y^2 - (x^p - a))$ is an integrally closed domain with k algebraically closed in $\mathrm{Frac}(A)$ but the domain $A \otimes_k k(a^{1/p})$ is *not* integrally closed (ruling out a possible generalization of (ii) without separability).

3. Let A be an integrally closed domain, $F = \mathrm{Frac}(A)$, F'/F a finite extension, and $A' \subset F'$ a module-finite A-subalgebra such that $F' = \mathrm{Frac}(A')$ (which clearly exists; why?). Assume A' is finite *free* as an A-module (e.g., this holds if A is a PID, such as $\mathbf{Z}$ or $k[x]$). Let $B(x,y) = \mathrm{Tr}_{F'/F}(xy)$.

(i) Let $\mathbf{e} = \{e_i\}$, $\mathbf{e}' = \{e'_i\}$ be ordered A-bases of A'. Show $(B(e_i, e_j)) = T^{\mathrm{t}}(B(e'_i, e'_j))T$ where the matrix T converts $\mathbf{e}$-coordinates into $\mathbf{e}'$-coordinates. Deduce that $\mathrm{disc}_{\mathbf{e}}(A'/A) := \det(B(e_i, e_j)) \in A$ is independent of $\mathbf{e}$ up to $(A^\times)^2$, and is nonzero if F'/F is separable. In particular, $\mathrm{disc}_{\mathbf{e}}(A'/A)A$ in A is independent of $\mathbf{e}$, as is the element $\mathrm{disc}_{\mathbf{e}}(A'/A) \in A$ when $A = \mathbf{Z}$ (so we may write $\mathrm{disc}(A'/\mathbf{Z})$).

(ii) For $K = \mathbf{Q}(\sqrt{d})$ with a square-free $d \in \mathbf{Z} - \{0, 1\}$, prove $D_K := \mathrm{disc}(\mathscr{O}_K/\mathbf{Z})$ is equal to $4d$ if $d \equiv 2, 3 \bmod 4$ and is equal to d if $d \equiv 1 \bmod 4$. Deduce that $\mathscr{O}_K = \mathbf{Z}[(D_K + \sqrt{D_K})/2]$.

(iii) For a finite extension K of $\mathbf{Q}$ and an order $\mathscr{O} \subset \mathscr{O}_K$ (see Exercise 1), prove that $\mathrm{disc}(\mathscr{O}/\mathbf{Z}) = [\mathscr{O}_K : \mathscr{O}]^2 \mathrm{disc}(\mathscr{O}_K/\mathbf{Z})$. Deduce that if $\mathrm{disc}(\mathscr{O}/\mathbf{Z})$ is squarefree then $\mathscr{O} = \mathscr{O}_K$! As an application, prove that for $K = \mathbf{Q}(\alpha)$ with $\alpha^3 - \alpha + 1 = 0$, $\mathscr{O}_K = \mathbf{Z}[\alpha]$ with $\mathrm{disc}(\mathscr{O}_K/\mathbf{Z}) = -23$.

4. Let $\varphi : A \to B$ be a map of rings. For radical $I \subset A$, $J \subset B$, prove $X(\varphi) : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ carries $V(J)$ into $V(I)$ if and only if $\varphi(I) \subset J$. In such cases, check the induced map $\overline{\varphi} : A/I \to B/J$ is compatible with the induced continuous map $V(J) \to V(I)$ (arising from $X(\varphi)$) in the sense that this latter is $X(\overline{\varphi})$ via the natural identifications of $V(J)$ with $\mathrm{Spec}(B/J)$ and $V(I)$ with $\mathrm{Spec}(A/I)$.

5. Read the handout on artinian rings and modules, noting the remarkable fact that artinian commutative rings are noetherian.

(i) For artinian $A \neq 0$, prove $\mathrm{Spec}(A)$ is finite and discrete.

(ii) If A is noetherian, nonzero, and all prime ideals are maximal and minimal (so there are only finitely many maximal ideals) then prove $X := \mathrm{Spec}(A)$ is a finite discrete space and that the natural map $A \to \prod_{x \in X} A_{\mathfrak{m}_x}$ (with $\mathfrak{m}_x$ the maximal ideal associated to x) is an isomorphism with each $A_{\mathfrak{m}_x}$ having a nilpotent maximal ideal. (Hint: adapt our earlier argument for A finite-dimensional over a field). Conclude that A is artinian!

(iii) If A is a nonzero finitely generated algebra over an arbitrary field k, show A is artinian if and only if $\dim_k A < \infty$. (Hint: for "$\Rightarrow$", reduce to the case of local A with nilpotent maximal ideal $\mathfrak{m}$ and recall that $[A/\mathfrak{m} : k] < \infty$ by the Nullstellensatz.)