

MATH 210B. HOMEWORK 7

1. For an integrally closed domain A and $F = \text{Frac}(A)$, let F'/F be an algebraic extension that is normal (i.e., all F -embeddings $F' \hookrightarrow \bar{F}$ have the same image). Let A' be the integral closure of A in F' . We claim $\text{Aut}(F'/F)$ acts transitively on fibers of $\text{Spec}(A') \rightarrow \text{Spec}(A)$.

(i) Assume F'/F is finite. If $K \subset F'$ is the (Galois) separable closure of F in F' , and B is the integral closure of A in K , show that $\text{Spec}(A') \rightarrow \text{Spec}(B)$ is bijective and even a homeomorphism. Use this to reduce to treating K instead of F' , so for F'/F finite we may assume F'/F is *Galois*.

(ii) Now with F'/F finite Galois, let $G = \text{Gal}(F'/F)$. Suppose $P, Q \in \text{Spec}(A')$ over $\mathfrak{p} \in \text{Spec}(A)$ are not in the same G -orbit. Pass to $A_{\mathfrak{p}}$ so \mathfrak{p} is maximal and hence P, Q are maximal. Use the Chinese Remainder Theorem to find $x \in A'$ so that $x \equiv 1 \pmod{g(P)}$ for all $g \in G$ but $x \equiv 0 \pmod{Q}$. Show that $N_{F'/F}(x) \in \mathfrak{p}$ (using $A' \cap F = A$!) yet $\prod_{g \in G} g(x) \notin P$, and obtain a contradiction.

(iii) For any F'/F , conclude by Zorn's Lemma. (Hint: for F -normal $E \subset F'$ show $\text{Aut}(F'/F) \twoheadrightarrow \text{Aut}(E/F)$ by working inside \bar{F} , and for $\alpha \in F'$ find finite $L/E(\alpha)$ normal over F inside F' .)

2. (i) For nonzero ideals \mathfrak{a} and \mathfrak{b} of a Dedekind domain R , prove that the multiplication map $\mathfrak{a} \otimes_R \mathfrak{b} \rightarrow \mathfrak{a}\mathfrak{b}$ is an isomorphism. (Hint: localize to the case of discrete valuation rings R .)

(ii) For a field k , let $R = k[x, y]/(y^2 - x^3)$ (a 1-dimensional noetherian domain, *not* Dedekind since $y/x \notin R$ but $(y/x)^2 = x \in R$). For the maximal ideal $\mathfrak{m} = (x, y) \subset R$, prove the R -linear map $\mathfrak{m} \otimes_R \mathfrak{m} \rightarrow \mathfrak{m}^2$ is *not* an isomorphism (hint: apply “reduction modulo \mathfrak{m} ”).

3. Use closedness properties of Spec with respect to module-finite ring maps to make the geometric proof of Noether Normalization over an algebraically closed field (using linear projections) work over any infinite field. (Don't just rewrite the argument in terms of pure commutative algebra. Embrace the geometric language of Spec and the Zariski topology, combined with algebra.)

4. Let A be a finitely generated algebra over a (possibly finite) field k .

(i) Prove $\dim(A \otimes_k K) = \dim(A)$ for *any* extension field K/k . (Hint: Using closed subsets of Spec , reduce to A a domain and use Noether Normalization; note $A \otimes_k K$ may not be a domain!)

(ii) Assume A is a domain. By the codimension handout, *every* maximal chain of irreducible closed sets in $X = \text{Spec}(A)$ has length $1 + \dim X$. Relate irreducible closed sets in $\text{Spec}(A_{\mathfrak{p}})$ to irreducible closed sets in X containing $\{\mathfrak{p}\}$, and deduce that $\dim A_{\mathfrak{p}} = \dim A - \dim(A/\mathfrak{p})$ ($< \infty$!).

5. We know that for a field F , the proper irreducible closed sets in $\text{Spec } F[x, y]$ that are not points are precisely “irreducible plane curves”: $\text{Spec } F[x, y]/(f)$ for irreducible $f \in F[x, y]$.

(i) Draw a picture of the map $\text{Spec } k[x, y, t] \rightarrow \text{Spec } k[t]$ for an algebraically closed field k , indicating the situation in the fiber over the generic point $\text{Spec } k(t)$ as well as the fibers over closed points $\text{Spec}(k[t]/(t - c)) = \text{Spec } k$ ($c \in k$). Within your picture, assuming $\text{char}(k) \neq 2$, draw $\text{Spec } k[x, y, t]/(y^2 - x^2 - h(t))$ for a non-constant $h \in k[t]$, indicating the different behavior over $t = c$ depending on whether or not $h(c) = 0$. Where are you using $\text{char}(k) \neq 2$?

(ii) Now replace $k[t]$ with \mathbf{Z} : draw an analogous picture for $\text{Spec } \mathbf{Z}[x, y] \rightarrow \text{Spec } \mathbf{Z}$, indicating the fiber over the generic point $\text{Spec } \mathbf{Q}$ as well as over a couple of closed points $\text{Spec } \mathbf{F}_p$. Within your picture, draw $\text{Spec } \mathbf{Z}[x, y]/(y^2 - x^2 - 15)$, indicating the different behavior for the fibers over $p = 3, 5$ versus over (0) or $p > 5$. The fiber over $p = 2$ is usually drawn as a “doubled line”; why?

6. Let A be Dedekind (so $A/\mathfrak{m}^e \simeq A_{\mathfrak{m}}/\mathfrak{m}^e A_{\mathfrak{m}}$ with $A_{\mathfrak{m}}$ a dvr for maximal $\mathfrak{m} \subset A$), $e_1, \dots, e_n \geq 0$.

(i) (weak approximation) For distinct maximal $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ find $a \in A - \{0\}$ satisfying $\text{ord}_{\mathfrak{p}_i}(a) = e_i$ for all i . (Since $\mathfrak{p}_j A_{\mathfrak{p}_i} = A_{\mathfrak{p}_i}$ for $j \neq i$, this says \mathfrak{p}_i as a factor of aA has multiplicity e_i .) Hint: CRT.

(ii) Using (i), show every nonzero proper ideal \mathfrak{a} in A is generated by 2 elements. (Hint: Let $\{\mathfrak{p}_i\}$ be the set of prime factors of \mathfrak{a} , with multiplicity e_i for \mathfrak{p}_i . Make $a \in A - \{0\}$ with $\text{ord}_{\mathfrak{p}_i}(a) = e_i$ for all i , and then $b \in A - \{0\}$ so that (a, b) has *only* the \mathfrak{p}_i as prime factors, with multiplicities e_i .)