Math 210B. Homework 8

1. (i) Compute the prime factorizations of (2) and (5) in $\mathbf{Z}[\sqrt{-6}]$, and (2) and (3) in $\mathbf{Z}[\sqrt{-5}]$, and check your work by multiplying out the products when (p) is not already prime.

(ii) Show $\mathfrak{p} := (3, 1+\sqrt{-5})$ is prime in $A = \mathbf{Z}[\sqrt{-5}]$ with $A/\mathfrak{p}$ of size 3, and that $A/(\alpha)$ has size $\mathrm{N}(\alpha) = a^2 + 5b^2$ for nonzero $\alpha = a + b\sqrt{-5}$ ($a, b \in \mathbf{Z}$). Deduce that $\mathfrak{p}$ is not principal, and find $\alpha \in A$ whose prime factorization has $\mathfrak{p}$ occurring once and $\overline{\mathfrak{p}} = (3, 1-\sqrt{-5})$ not occurring; compute the prime factorization of (α) explicitly. (Recall that $\mathfrak{m}|(\alpha)$ when $\alpha \in \mathfrak{m}$.)

2. Using weak approximation, show any Dedekind domain with finitely many maximal ideals is a PID. Using that $\mathbf{Z}[\sqrt{-6}]$ is not a UFD, deduce $\mathbf{Z}$ has infinitely many primes!

3. Let G be an arbitrary group (not necessarily finite), and consider arbitrary linear representations of G (not necessarily finite-dimensional) over any field k.

(i) If (W, ρ) is a G-representation, show its dual W^* is a G-representation via $g.\ell = \ell \circ \rho(g^{-1})$ (especially that this is a *left* G-action on W^*). Prove this is the unique action making evaluation $W \otimes W^* \to k$ be G-equivariant when using the trivial action on the target k.

(ii) Let $B : W \times W' \to k$ be a bilinear pairing between representation spaces. Prove that B is G-invariant in the sense that $B(gw, gw') = B(w, w')$ for all w, w', g if and only if the induced linear map $W \otimes W' \to k$ is G-equivariant (using the trivial action on k), and also if and only if the induced linear map $W' \to W^*$ (via $w' \mapsto B(\cdot, w')$) is G-equivariant. (For example, by (i), $W \to W^{**}$ is G-equivariant.)

(iii) Define natural G-actions on $\mathrm{Sym}^n(V)$ and $\wedge^n(V)$ ($n \geq 1$), and if $\dim V < \infty$ prove the natural isomorphisms $(V^*)^{\otimes n} \simeq (V^{\otimes n})^*$ and $\wedge^n(V^*) \simeq (\wedge^n V)^*$ are G-equivariant.

4. Let G be an arbitrary group (not necessarily finite) and (V, ρ) and (V', ρ') be arbitrary representation spaces (not necessarily finite-dimensional) over an arbitrary field k. Define a G-action on the k-vector space $\mathrm{Hom}_k(V, V')$ by $g.T = \rho'(g) \circ T \circ \rho(g^{-1})$.

(i) Prove this really is a G-representation structure (in particular, a *left* G-action) on the k-vector space $\mathrm{Hom}_k(V, V')$ and that $\mathrm{Hom}_k(V, V')^G = \mathrm{Hom}_{k[G]}(V, V')$.

(ii) If $\dim V < \infty$, prove the natural isomorphism $V' \otimes V^* \simeq \mathrm{Hom}_k(V, V')$ is G-equivariant.

5. Let $G = D_n := \langle \sigma, \tau \,|\, \tau^2 = 1, \sigma^n = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ (dihedral of order $2n$) with $n \geq 3$.

(i) For $\zeta \in \mathbf{C}^\times$ an nth root of unity distinct from ± 1 (so $\zeta \notin \mathbf{R}$), let ρ_ζ be the unique representation of G on $V = \mathbf{C}^2$ defined by

$$\rho_\zeta(\sigma) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad \rho_\zeta(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

check the relations hold so that such a ρ_ζ really exists. Prove directly that ρ_ζ is irreducible (what line can be σ-stable?), and if $\overline{V}$ denotes the $\mathbf{C}$-linear representation obtained by scalar extension through complex conjugation $\mathbf{C} \simeq \mathbf{C}$ then construct a $\mathbf{C}$-*linear* isomorphism $\overline{V} \simeq V$ of G-representations via an explicit $m \in \mathrm{GL}_2(\mathbf{C})$ conjugating ρ_ζ into $\overline{\rho_\zeta} = \rho_{\zeta^{-1}}$.

(ii) Find an explicit matrix in $\mathrm{GL}_2(\mathbf{C})$ that conjugates $\rho_{e^{2\pi i/n}}$ to the standard representation of D_n in $\mathrm{GL}_2(\mathbf{R})$ with σ acting as clockwise rotation of angle $2\pi/n$ and τ acting as $\mathrm{diag}(1, -1)$, so $V \simeq \mathbf{C} \otimes_{\mathbf{R}} V_0$ for an $\mathbf{R}[G]$-module V_0. Deduce $\mathrm{End}_{\mathbf{R}[G]}(V_0) = \mathbf{R}$ (hint: show $k' \otimes_k \mathrm{End}_{k[H]}(W) = \mathrm{End}_{k'[H]}(k' \otimes_k W)$ for any field extension k'/k, finite group H, and finite-dimensional H-representation W over k) and that $\mathrm{End}_{\mathbf{R}[G]}(V) \simeq \mathrm{Mat}_2(\mathbf{R})$.