Math 210B. Homework 9

1. Let (V_i, ρ_i) $(i = 1, 2)$ be finite-dimensional representations of a finite group G_i over a field k with $\mathrm{char}(k) = 0$. Let $G = G_1 \times G_2$ and make $V = V_1 \otimes_k V_2$ a G-representation space via $\rho(g_1, g_2) : v_1 \otimes v_2 \mapsto (g_1.v_1) \otimes (g_2.v_2)$. Let $\chi_i = \chi_{\rho_i} : G_i \to k$ and $\chi = \chi_\rho : G \to k$.

(i) Prove $\langle \chi, \chi \rangle_G = \langle \chi_1, \chi_1 \rangle_{G_1} \cdot \langle \chi_2, \chi_2 \rangle_{G_2}$. When $k = \overline{k}$, prove χ is irreducible for G if and only if each χ_i is irreducible for G_i. Give a counterexample to "if" with $k = \mathbf{R}$ (hint: cyclic $G_1 = G_2 = H$, $V_1 = V_2$ the underlying $\mathbf{R}$-representation of $\chi : H \to \mathbf{C}^\times$ of order > 2).

(ii) Assume $k = \overline{k}$. Prove χ determines the χ_i's, and that every irreducible representation of $G_1 \times G_2$ over k arises this way. (Hint: View a $G_1 \times G_2$-representation as one for each G_i separately. For the second part, if V_1 is an irreducible G_1-subrepresentation of V then view the "multiplicity space" $\mathrm{Hom}_{k[G_1]}(V_1, V)$ as a nonzero G_2-representation and consider an irreducible G_2-subrepresentation V_2; is evaluation $V_1 \otimes_k V_2 \to V$ nonzero and G-equivariant?)

2. Let G be a finite group. A finite-dimensional G-representation V over a field k (allowing $\mathrm{char}(k) | \#G$) is called *absolutely irreducible* if $V_{\overline{k}} := \overline{k} \otimes_k V$ is irreducible over $\overline{k}$.

(i) If $\#G \in k^\times$, show V is absolutely irreducible $\Leftrightarrow \mathrm{End}_G(V) = k$, and $\Leftrightarrow \langle \chi_V, \chi_V \rangle_G = 1$ when $\mathrm{char}(k) = 0$. Find some G and irreducible $\mathbf{R}[G]$-module W so that $W_{\mathbf{C}}$ is reducible.

(ii) If $k = \overline{k}$ show for every irreducible G-representation V over k and extension field K/k that V_K is irreducible as a G-representation over K. (Hint: to argue by contradiction if V_K has a nonzero proper subrepresentation U, reduce to K/k finitely generated, "spread out" over a big finitely generated k-subalgebra $A \subset K$, and reduce modulo a maximal ideal of A!)

(iii) Using $\overline{\mathbf{Q}}[G]$ and $\mathbf{C}[G]$, prove via (ii) that every irreducible G-representation W over $\mathbf{C}$ arises over a finite extension $F/\mathbf{Q}$. Adapt to characteristic $p \nmid \#G$ for descent to finite fields. Bonus points if you handle $p | \#G$. [In char. 0 obviously F contains the field $\mathbf{Q}(\chi_W)$ generated by $\chi_W(G)$, but $F = \mathbf{Q}(\chi_W)$ may not work; the obstruction uses class field theory. Clearly $\mathbf{Q}(\chi_W) \subset \mathbf{Q}(\zeta_n)$ for the exponent n of G, and Brauer proved $\mathbf{Q}(\zeta_n)$ always works!]

3. (i) Let A be an associative ring, M a right A-module, and N a left A-module. Construct an abelian group $M \otimes_A N$ universal for bi-additive $B : M \times N \to P$ to abelian groups P such that $B(ma, n) = B(m, an)$. If $A \to A'$ is a ring map, how is $A' \otimes_A N$ a left A'-module?

(ii) Let G be *any* group, H a subgroup, k *any* field. For a left $k[H]$-module V, the *compact induction* $\text{c-Ind}_H^G(V)$ is the space of set maps $f : G \to V$ satisfying $f(hg) = h.f(g)$ ($h \in H$, $g \in G$) and vanishing outside *finitely many* cosets Hg_i; this is a $k[G]$-module via $g.f : x \mapsto f(xg)$. Check $g.f \in \text{c-Ind}_H^G(V)$ and $g'.(g.f) = (g'g).f$, and $k[G] \otimes_{k[H]} V \simeq \text{c-Ind}_H^G(V)$ as $k[G]$-modules by assigning to $[g] \otimes v$ the map $f : G \to V$ sending hg^{-1} to $h.v$ and vanishing off $Hg^{-1} \in H\backslash G$. Show the inverse is $f \mapsto \sum_{\overline{g} \in G/H} [g] \otimes f(g^{-1})$ (why a finite well-posed sum?). If $[G : H], \dim_k V < \infty$, show $\dim_k \text{c-Ind}_H^G(V) = [G : H] \dim_k V$ and give a k-basis.

(iii) Build $\theta : \mathrm{Hom}_G(\text{c-Ind}_H^G(V), W) \simeq \mathrm{Hom}_H(V, W)$ functorial in $k[G]$-modules W (*Frobenius reciprocity*), compute the inverse explicitly, and show $\text{c-Ind}_H^G(1) \simeq \oplus_{\overline{g} \in G/H} k[\overline{g}]$.

4. For G, H, k, V as in 3(ii), the *induction* $\mathrm{Ind}_H^G(V)$ is the $k[G]$-module defined like $\text{c-Ind}_H^G(V)$ but without vanishing off finitely many cosets in $H\backslash G$ (so they coincide when $[G : H]$ is finite; e.g., finite G). Show $\mathrm{Hom}_G(W, \mathrm{Ind}_H^G(V)) \simeq \mathrm{Hom}_H(W, V)$ functorial in $k[G]$-modules W (*Frobenius reciprocity*) by composing with evaluation $\mathrm{ev}_1 : \mathrm{Ind}_H^G(V) \to V$ at 1. (Hint: show $f \mapsto (w \mapsto (g \mapsto f(gw)))$ is a well-defined inverse.) What is $\mathrm{Ind}_H^G(1)$?