As an application of the Conjugacy Theorem, we can describe the continuous class functions \( f : G \to \mathbb{C} \) on a connected compact Lie group \( G \) in terms of a choice of maximal torus \( T \subset G \). This will be an important “Step 0” in our later formulation of the Weyl character formula. If we consider \( G \) acting on itself through conjugation, the quotient \( \text{Conj}(G) \) by that action is the space of conjugacy classes. We give it the quotient topology from \( G \), so then the \( \mathbb{C} \)-algebra of continuous \( \mathbb{C} \)-valued class functions on \( G \) is the same as the \( \mathbb{C} \)-algebra \( C^0(\text{Conj}(G)) \) of continuous \( \mathbb{C} \)-valued class functions on \( \text{Conj}(G) \).

Let \( W = N_G(T)/T \) be the (finite) Weyl group, so \( W \) naturally acts on \( T \). The \( W \)-action on \( T \) is induced by the conjugation action of \( N_G(T) \) on \( G \), so we get an induced continuous map of quotient spaces \( T/W \to \text{Conj}(G) \).

**Proposition 0.1.** The natural continuous map \( T/W \to \text{Conj}(G) \) is bijective.

**Proof.** By the Conjugacy Theorem, every element of \( G \) belongs to a maximal torus, and such tori are \( G \)-conjugate to \( T \), so surjectivity is clear. For injectivity, consider \( t, t' \in T \) that are conjugate in \( G \). We want to show that they lie below to the same \( W \)-orbit in \( T \).

Pick \( g \in G \) so that \( t' = gtg^{-1} \). The two tori \( T, gTg^{-1} \) then contain \( t' \), so by connectedness and commutativity of tori we have \( T, gTg^{-1} \subset Z_G(t')^0 \). But these are maximal tori in \( Z_G(t')^0 \) since they’re even maximal in \( G \), and \( Z_G(t')^0 \) is a connected compact Lie group. Hence, by the Conjugacy Theorem applied to this group we can find \( z \in Z_G(t')^0 \) such that \( z(gTg^{-1})z^{-1} = T \). That is, \( zg \) conjugates \( T \) onto itself, or in other words \( zg \in N_G(T) \). Moreover,

\[
(zg)t(zg)^{-1} = z(gtg^{-1})z^{-1} = zt'z^{-1} = t',
\]

the final equality because \( z \in Z_G(t') \). Thus, the class of \( zg \) in \( W = N_G(T)/T \) carries \( t \) to \( t' \), as desired. \[ \blacksquare \]

To fully exploit the preceding result, we need the continuous bijection \( T/W \to \text{Conj}(G) \) to be a homeomorphism. Both source and target are compact spaces, so to get the homeomorphism property we just need to check that each is Hausdorff. The Hausdorff property for these is a special case of:

**Lemma 0.2.** Let \( X \) be a locally compact Hausdorff topological space equipped with a continuous action by a compact topological group \( H \). The quotient space \( X/H \) with the quotient topology is Hausdorff.

**Proof.** This is an exercise in definitions and point-set topology. \[ \blacksquare \]

Combining this lemma with the proposition, it follows that the \( \mathbb{C} \)-algebra \( C^0(\text{Conj}(G)) \) of continuous \( \mathbb{C} \)-valued class functions on \( G \) is naturally identified with \( C^0(T/W) \), and it is elementary (check!) to identify \( C^0(T/W) \) with the the \( \mathbb{C} \)-algebra \( C^0(T)^W \) of \( W \)-invariant continuous \( \mathbb{C} \)-valued functions on \( T \). Unraveling the definitions, the composite identification

\[
C^0(\text{Conj}(G)) \simeq C^0(T)^W
\]

of the \( \mathbb{C} \)-algebras of continuous \( \mathbb{C} \)-valued class functions on \( G \) and \( W \)-invariant continuous \( \mathbb{C} \)-valued functions on \( T \) is given by \( f \mapsto f|_T \).