Math 210C. Examples of maximal compact subgroups

1. Introduction

It is a general fact (beyond the scope of this course) that if G is a Lie group with finitely many connected components then: every compact subgroup of G is contained in a maximal one (i.e., one not strictly contained in a larger compact subgroup), all maximal compact subgroups $K \subset G$ are G-conjugate to each other, and K meets every connected component of G with $G^0 \cap K$ connected and itself a maximal compact subgroup of G^0. Nearly all treatments of the story of maximal compact subgroups of Lie groups in textbooks only address the connected case, but Hochschild's book "Structure of Lie Groups" (see Chapter XV, Theorem 3.1) does treat the wider case with $\pi_0(G)$ merely finite (possibly not trivial); I don't think one can deduce the case of finite $\pi_0(G)$ from the case of trivial $\pi_0(G)$. Cases with $\pi_0(G)$ finite but possibly non-trivial do arise very naturally as the group of $\mathbf{R}$-points of affine group varieties over $\mathbf{R}$.

In this handout, we address a few classes of examples for which such K and their conjugacy can be verified directly. Some aspects of the technique used below actually play an essential role in the treatment of the general case (but we don't have time to get into that, so we refer the interested reader to Hochschild's book for such further details).

2. The definite cases

The basic building blocks for everything below emerge from two cases: $\mathrm{GL}(V)$ for a finite-dimensional nonzero vector space V over $\mathbf{R}$ and $\mathrm{GL}(W)$ for a finite-dimensional nonzero vector space W over $\mathbf{C}$. We know by Gram-Schmidt that $\mathrm{GL}(V)$ acts transitively on the set of all positive-definite (non-degenerate) quadratic forms q on V (this just expresses that all such q become "the same" in suitable linear coordinates), so the compact subgroups $\mathrm{O}(q)$ of $\mathrm{GL}(V)$ constitute a single conjugacy class. Likewise, $\mathrm{GL}(W)$ acts transitively on the set of all positive-definite (non-degenerate) hermitian forms h on W (this just expresses that all such h become "the same" in suitable $\mathbf{C}$-linear coordinates), so the compact subgroups $\mathrm{U}(h)$ of $\mathrm{GL}(W)$ constitute a single conjugacy class.

We claim that every compact subgroup of $\mathrm{GL}(V)$ lies in some $\mathrm{O}(q)$, and every compact subgroup of $\mathrm{GL}(W)$ lies in some $\mathrm{U}(h)$. It is elementary to check that a compact Lie group (so finite π_0; why?) has no proper closed C^∞-submanifold of the same dimension with the same number of connected components (why not?). Hence, it would follow that the compact subgroups $\mathrm{O}(q) \subset \mathrm{GL}(V)$ are all maximal and likewise for the compact subgroups $\mathrm{U}(h) \subset \mathrm{GL}(W)$, since we know from HW1 that such inclusions are closed C^∞-submanifolds.

Remark 2.1. As an exercise, the interested reader can then deduce similar results for $\mathrm{SL}(V)$ and $\mathrm{SL}(W)$ using $\mathrm{SO}(q)$ and $\mathrm{SU}(h)$ (retaining conjugacy properties because $\mathrm{O}(q) \cdot \mathbf{R}_{>0} \to \mathrm{GL}(V)/\mathrm{SL}(V)$ and $\mathrm{U}(h) \cdot \mathbf{R}_{>0} \to \mathrm{GL}(W)/\mathrm{SL}(W)$ are surjective, where $\mathbf{R}_{>0}$ is the evident central subgroup of such scalars in each case).

So let K be a compact subgroup of $\mathrm{GL}(V)$. We seek a positive-definite $q : V \to \mathbf{R}$ such that $K \subset \mathrm{O}(q)$. Choose a positive-definite inner product $\langle \cdot, \cdot \rangle_0$ on V. We want to make a new one that is K-invariant by averaging. If K is a finite group then this can be done as a genuine average: make a new bilinear form

$$\langle v, w\rangle = \frac{1}{\#K}\sum_{k\in K}\langle kv, kw\rangle_0.$$

(also works if we omit the scaling factor $1/\#K$). This is manifestly K-invariant by design, and positive-definite. In the more meaty case that K is not finite (but compact), one has to use a *Haar measure* on K. So let us briefly digress to record the basic existence/uniqueness results on Haar measures (which for Lie groups we will later build via differential forms).

If G is any locally compact Hausdorff group, a *left Haar measure* is a regular Borel measure μ on the topological space G with the invariance property $\mu(gA) = \mu(A)$ for all Borel sets $A \subset G$ and all $g \in G$. (If we use Ag then we speak of a "right Haar measure".) For example, if $G = \mathbf{R}^n$ then the Lebesgue measure is a left Haar measure. As another example, if G is a discrete group (e.g., a finite group with the discrete topology) then *counting measure* m_G (i.e., $m_G(A) = \#A \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$) is a left Haar measure. The "regularity" condition in the definition of a Haar measure is a technical property which avoids some pathologies, and imposes in particular that $\mu(U) > 0$ for non-empty open subsets $U \subset G$ and $\mu(C) < \infty$ for compact subsets $C \subset G$. (For example, counting measure m_G on any G is a translation-invariant Borel measure but if G is non-discrete then there exist infinite compact $C \subset G$ and $m_G(C) = \infty$, so m_G is not regular and thus not a Haar measure if G is non-discrete.)

The basic result about left Haar measures μ is that they exist and are unique up to an $\mathbf{R}_{>0}$-scaling factor (and likewise for right Haar measures). For Lie groups we will construct them using differential forms. For many interesting groups the left Haar measures are also right Haar measures, in which case we call G *unimodular*. We'll later show that compact groups are unimodular (as are important non-compact Lie groups such as $\mathrm{SL}_n(\mathbf{R})$, $\mathrm{Sp}_{2n}(\mathbf{C})$, and so on, but we will not need this). In case G is *compact*, regularity implies $\mu(G)$ is both finite and positive, so we can scale μ by $1/\mu(G)$ to arrive at μ satisfying $\mu(G) = 1$. This "normalized" property removes all scaling ambiguity and so pins down a canonical (left) Haar measure in the compact case, denoted μ_G. For example, if G is finite then the normalized (left) Haar measure is $\mu_G(A) = \#A/\#G$; i.e., μ_G assigns mass $1/\#G$ to each element of G.

Coming back to a compact subgroup K of $\mathrm{GL}(V)$, we use Haar measures to prove:

Lemma 2.2. *There exists a K-invariant positive-definite inner product $\langle \cdot, \cdot\rangle$ on V.*

Proof. Pick a positive-definite inner product $\langle \cdot, \cdot\rangle_0$. We will make a K-invariant one by averaging that initial choice in the sense of integration over K against a right Haar measure μ_K (which happens to also be a left Haar measure, though we don't need that):

$$\langle v, w\rangle = \int_K \langle kv, kw\rangle_0 \,\mathrm{d}\mu_K.$$

(The integrand is a continuous function of k, so the integral makes sense and converges since K is compact.) This new pairing is certainly bilinear and *positive-definite* (why?), and it is K-invariant precisely because μ_K is a right Haar measure (replacing v and w with $k_0 v$ and $k_0 w$ respectively for some $k_0 \in K$ amounts to translating the integrand through *right-translation* against k_0, so the right-invariance of μ_K ensures the integral is unaffected by this intervention of k_0). ■

Remark 2.3. The preceding argument would run into difficulties if we tried to build a K-invariant non-degenerate symmetric bilinear form with an indefinite signature: the problem is that the integration construction would not have positive or negative-definiteness properties, and so we would not have a way to ensure the end result is non-degenerate (or even to control its signature).

This K-invariant inner product $\langle \cdot, \cdot \rangle$ corresponds to a K-invariant positive-definite quadratic form q on V, so $K \subset \mathrm{O}(q)$. That does the job for $\mathrm{GL}(V)$. In the case of $\mathrm{GL}(W)$, we can likewise build a K-invariant positive-definite hermitian form h on W (so $K \subset \mathrm{U}(h)$) by exactly the same K-averaging method beginning with an initial choice of positive-definite hermitian form on W. In particular, as with definite orthogonal groups, the compact subgroups $\mathrm{U}(h) \subset \mathrm{GL}(W)$ are maximal.

As as we noted in Remark 2.1, it follows immediately from our results for $\mathrm{GL}(V)$ and $\mathrm{GL}(W)$ that we get analogous results for $\mathrm{SL}(V)$ and $\mathrm{SL}(W)$ using $\mathrm{SO}(q)$ and $\mathrm{SU}(h)$ (with positive-definite q on V and h on W).

3. Indefinite orthogonal groups

Now we fix a quadratic form $q : V \to \mathbf{R}$ with signature (r, s) with $r, s > 0$, so $\mathrm{O}(q) \simeq \mathrm{O}(r, s)$ is non-compact. Let $n = r + s$. The technique from HW1 in the positive-definite case adapts with only minor changes in the indefinite case to yield that $\mathrm{O}(q)$ is a closed C^∞-submanifold of $\mathrm{GL}(V)$. Can we describe its maximal compact subgroups and see the conjugacy of all of them by a direct method? The answer is "yes", and the key input will be the spectral theorem, combined with the analysis of compact subgroups inside $\mathrm{GL}_m(\mathbf{R})$ in the previous section for any $m > 0$ (e.g., $m = r, s$). First we build a conjugacy class of compact subgroups of $\mathrm{O}(q)$, and then check that they are maximal and every compact subgroup lies in one of these.

Consider a direct-sum decomposition

$$V = V^+ \oplus V^-$$

for which $q|_{V^+}$ is positive-definite, $q|_{V^-}$ is negative-definite, and V^+ is B_q-orthogonal to V^-. Such decompositions exist precisely by the classification of non-degenerate quadratic spaces over $\mathbf{R}$: we know that in suitable linear coordinates q becomes $\sum_{j=1}^{r} x_j^2 - \sum_{j=r+1}^{n} x_j^2$, so we can take V^+ to be the span of the first r such basis vectors and V^- to be the span of the others in that basis.

For any such decomposition, choices of orthonormal bases for the positive-definite spaces (V^+, q) and $(V^-, -q)$ give a description of q as $\sum_{j=1}^{d} y_j^2 - \sum_{j=d+1}^{n} y_j^2$ for linear coordinates dual to the union of those two bases, with $d = \dim V^+$. Hence, $d = r$ by the well-definedness of signature, so $\dim V^+ = r$ and $\dim V^- = n - r = s$. We thereby get a compact subgroup

$$\mathrm{O}(V^+) \times \mathrm{O}(V^-) \subset \mathrm{O}(q)$$

that is a copy of $\mathrm{O}(r) \times \mathrm{O}(s)$. (Here we write $\mathrm{O}(V^\pm)$ to denote $\mathrm{O}(\pm q|_{V^\pm})$.) By using the closed C^∞-submanifold $\mathrm{GL}(V^+) \times \mathrm{GL}(V^-) \subset \mathrm{GL}(V)$ (visualized via block matrices), we see that $\mathrm{O}(V^+) \times \mathrm{O}(V^-)$ is a closed C^∞-submanifold of $\mathrm{GL}(V)$ which is contained in the closed C^∞-submanifold $\mathrm{O}(q) \subset \mathrm{GL}(V)$, so $\mathrm{O}(V^+) \times \mathrm{O}(V^-)$ is a closed C^∞-submanifold of $\mathrm{O}(q)$.

We claim that these subgroups constitute a single $\mathrm{O}(q)$-conjugacy class, and that these are all maximal, with every compact subgroup of $\mathrm{O}(q)$ contained in one of these. First we address conjugacy:

Lemma 3.1. *If* (V^+, V^-) *and* (U^+, U^-) *are two such pairs for* (V, q) *then each is carried to the other by an element of* $\mathrm{O}(q)$*.*

Proof. We have isomorphisms of quadratic spaces

$$(V^+, q|_{V^+}) \perp (V^-, q|_{V^-}) \simeq (V, q) \simeq (U^+, q|_{U^+}) \perp (U^-, q|_{U^-})$$

(where $(W', q') \perp (W'', q'')$ means $W' \oplus W''$ equipped with the quadratic form $Q(w' + w'') = q'(w') + q''(w'')$; it is easy to check that W' and W'' are B_Q-orthogonal with $Q|_{W'} = q'$ and $Q|_{W''} = q''$). But $(V^+, q|_{V^+})$ and $(U^+, q|_{U^+})$ are positive-definite quadratic spaces with the *same* dimension r, and likewise $(V^-, -q|_{V^-})$ and $(U^-, -q|_{U^-})$ are positive-definite quadratic spaces with the *same* dimension $n - r = s$.

By the Gram-Schmidt process, any two positive-definite quadratic spaces over $\mathbf{R}$ with the same finite dimension are isomorphic (as quadratic spaces), so there exist linear isomorphisms $T^\pm : V^\pm \simeq U^\pm$ that carry $\pm q|_{V^\pm}$ over $\pm q|_{U^\pm}$. Hence, the linear automorphism

$$V = V^+ \oplus V^- \overset{T^+ \oplus T^-}{\simeq} U^+ \oplus U^- = V$$

preserves q (why?) and carries $V^\pm$ over to $U^\pm$. This is exactly an element of $\mathrm{O}(q) \subset \mathrm{GL}(V)$ carrying the pair (V^+, V^-) over to (U^+, U^-). ∎

We conclude from the Lemma that the collection of compact subgroups $\mathrm{O}(V^+) \times \mathrm{O}(V^-) \subset \mathrm{O}(q)$ (that are also compact Lie groups) constitute a single $\mathrm{O}(q)$-conjugacy class. Thus, exactly as in the treatment of the definite case, considerations of dimension and finiteness of π_0 imply that such subgroups are maximal provided that we can show *every* compact subgroup $K \subset \mathrm{O}(q)$ is contained in one of these.

In other words, for a given K, our task reduces to finding an ordered pair (V^+, V^-) as above that is stable under the K-action on V. To achieve this, we will use the spectral theorem over $\mathbf{R}$. We first choose a K-invariant positive-definite inner product $\langle \cdot, \cdot \rangle$ on V, as we have already seen can be done (in effect, this is just applying our knowledge about the maximal compact subgroups of $\mathrm{GL}(V)$ and that every compact subgroup of $\mathrm{GL}(V)$ is contained in one of those). Let's use this inner product to identify V with its own dual. For our indefinite non-degenerate quadratic form $q : V \to \mathbf{R}$, consider the associated symmetric bilinear form $B_q : V \times V \to \mathbf{R}$ that is a perfect pairing (by non-degeneracy of q). This gives rise to a linear isomorphism $T_q : V \simeq V^*$ via $v \mapsto B_q(v, \cdot) = B_q(\cdot, v)$ which is self-dual (i.e., equal to its own dual map via double-duality on V) due to the symmetry of B_q.

Composing T_q with the self-duality $\iota : V^* \simeq V$ defined by the choice of $\langle \cdot, \cdot \rangle$, we get a composite linear isomorphism

$$f : V \overset{T_q}{\simeq} V^* \simeq V;$$

explicitly, $f(v)$ is the unique element of V such that $B_q(v, \cdot) = \langle f(v), \cdot \rangle$ in V^*. The crucial observation is that f is *self-adjoint* with respect to $\langle \cdot, \cdot \rangle$ (i.e., f is self-dual relative to ι); this is left to the interested reader to check as an exercise (do check it!). Thus, by the spectral theorem, f is diagonalizable. The eigenvalues of f are nonzero since f is an isomorphism.

(Typically f will have very few eigenvalues, with multiplicity that is large.) But $\langle\cdot,\cdot\rangle$ is K-invariant by design, and B_q is K-invariant since $K \subset \mathrm{O}(q)$, so f commutes with the K-action (i.e., $k.f(v) = f(k.v)$ for all $k \in K$ and $v \in V$). Hence, K must preserve each eigenspace of f. For v in the eigenspace V_λ for a given eigenvalue λ of f, we have

$$2q(v) = B_q(v,v) = \langle f(v), v\rangle = \langle \lambda v, v\rangle = \lambda\langle v, v, \rangle = \lambda\|v\|^2$$

where $\|\cdot\|$ is the norm associated to the positive-definite $\langle\cdot,\cdot\rangle$. Hence, $q|_{V_\lambda}$ is *definite* with sign equal to that of λ.

If V^+ denotes the span of the eigenspaces for the positive eigenvalues of f and V^- denotes the span of the eigenspaces for the negative eigenvalues of f then $q|_{V^\pm}$ is definite with sign $\pm$, each of $V^\pm$ are K-stable, and $V^+ \oplus V^- = V$. Hence, (V^+, V^-) is exactly an ordered pair of the desired type which is K-stable!

It follows similarly to the cases of $\mathrm{SL}(V)$ that for indefinite q on V with signature (r,s) that $\mathrm{SO}(q)$ has as its maximal compact subgroups exactly the disconnected compact groups

$$\{(g,g') \in \mathrm{O}(V^+) \times \mathrm{O}(V^-)) \mid \det(g) = \det(g')\} \simeq \{(T,T') \in \mathrm{O}(r) \times \mathrm{O}(s) \mid \det(T) = \det(T')\}$$

(and that every compact subgroup of $\mathrm{SO}(q)$ lies in one of these).

4. Indefinite unitary groups

Now we consider a complex vector space W with positive finite dimension n and an indefinite non-degenerate hermitian form h on W of type (r,s) with $0 < r < n$ and $s = n - r$. As in the indefinite orthogonal case, the technique from HW1 for positive-definite hermitian forms adapts to show that $\mathrm{U}(h)$ is a closed C^∞-submanifold of $\mathrm{GL}(W)$.

Inside $\mathrm{U}(h)$ there are compact subgroups $\mathrm{U}(W^+) \times \mathrm{U}(W^-) \simeq \mathrm{U}(r) \times \mathrm{U}(s)$ for $W^\pm \subset W$ complementary subspaces on which h is definite with sign $\pm$ and that are h-orthogonal to each other. Similarly to the orthogonal case, these are closed C^∞-submanifolds of $\mathrm{U}(h)$ and necessarily $\dim_{\mathbf{C}} W^+ = r$ and $\dim_{\mathbf{C}} W^- = s$ with orthonormal bases of $(W^\pm, \pm h|_{W^\pm})$ yielding a description of h as

$$h(w,w') = \sum_{j=1}^{n} \varepsilon_j w_j \overline{w}'_j$$

where $\varepsilon_j = 1$ for $1 \le j \le r$ and $\varepsilon_j = -1$ for $r+1 \le j \le n$.

An argument similar to the orthogonal case shows that $\mathrm{U}(h)$ acts transitively on the set of such ordered pairs (W^+, W^-) (using there is only one isomorphism class of positive-definite hermitian spaces of a given finite dimension). The spectral theorem for self-adjoint operators over $\mathbf{C}$ (rather than over $\mathbf{R}$) then enables us to adapt the technique in the orthogonal case (exerting some extra attention to the intervention of conjugate-linearity) to deduce that any compact subgroup $K \subset \mathrm{U}(h)$ is contained in $\mathrm{U}(W^+) \times \mathrm{U}(W^-)$ for some such ordered pair (W^+, W^-). We have directly proved the expected results for compact and maximal compact subgroups of $\mathrm{U}(h)$ in the indefinite case akin to the case of orthogonal groups.

It follows similarly to the case of $\mathrm{SL}(W)$ that for indefinite h on W with type (r,s) that $\mathrm{SU}(h)$ has as its maximal compact subgroups exactly the connected compact groups

$$\{(g,g') \in \mathrm{U}(W^+) \times \mathrm{O}(W^-)) \mid \det(g) = \det(g')\} \simeq \{(T,T') \in \mathrm{U}(r) \times \mathrm{U}(s) \mid \det(T) = \det(T')\}$$

(and that every compact subgroup of $\mathrm{SU}(h)$ lies in one of these).