Math 216A. Valuative criteria

1. Introduction

Let $f : X \to Y$ be a morphism of schemes. This handout proves "functor of points" criteria of Grothendieck (suggested by a notion called *completeness* in Weil's Foundations) for f to be separated or universally closed or proper. These are stated in terms of morphisms from spectra of valuation rings; see [Mat, §10] for a nice self-contained development of the basics of valuation rings (as a generalization of discrete valuation rings).

To give some motivation for what we will prove, for a field k consider the formal power series ring $k[\![t]\!]$ in 1 variable over k (a discrete valuation ring with the t-adic valuation). The ring $k(\!(t)\!) = k[\![t]\!][1/t]$ is its fraction field (the finite-tailed "formal Laurent series" over k). There is a useful principle in algebraic geometry that $\mathrm{Spec}(k[\![t]\!])$ is analgous to the open unit disc $\Delta = \{t \in \mathbf{C} \mid |t| < 1\}$ and the open generic point $\mathrm{Spec}(k(\!(t)\!))$ obtained by removing the closed point is analogous to the punctured open unit disc $\Delta^* = \Delta - \{0\}$.

If X is a topological space then by uniqueness of limits (of sequences, or of nets) in Hausdorff spaces we see that a *necessary* condition for X to be Hausdorff is that a continuous map $\Delta^* \to X$ has at most one extension to a continuous map $\Delta \to X$. On the other hand, by using some basic techniques in projective algebraic geometry together with Chow's Lemma it can be shown that when X is the "analytification" of a separated $\mathbf{C}$-scheme of finite type then a *sufficient* condition for X to be proper is that any holomorphic map $h : \Delta^* \to X$ extends to a holomorphic map $\Delta \to X$.

[This sufficient analytic compactness criterion is *not* necessary even for $X = \mathbf{CP}^1 \times \mathbf{CP}^1$ and h whose non-empty fibers have only 1 point, by considering $h(z) = (z, e^{1/z})$. And the "algebraic" nature of X cannot be omitted from a sufficient analytic compactness criterion in the Hausdorff case using maps from Δ^*, since by the removable singularities lemma in complex analysis any holomorphic map $\Delta^* \to \Delta$ extends to a holomorphic map $\Delta \to \Delta$ yet the target Δ is not compact.]

Much as separatedness and properness of a morphism $X \to Y$ are respectively analogues in the category of Y-schemes of the "Hausdorff" and "compact Hausdorff" conditions from topology, Grothendieck's *valuative criteria* for a Y-scheme X to be separated or proper over Y are expressed in terms of Y-maps

$$\mathrm{Spec}(K) \to X, \ \ \mathrm{Spec}(V) \to X$$

where V is a valuation ring for which a Y-morphism $\mathrm{Spec}(V) \to Y$ is given and K is the fraction field of V (so $\mathrm{Spec}(K)$ is a Y-scheme via $\mathrm{Spec}(K) \to \mathrm{Spec}(V) \to Y$). We regard $\mathrm{Spec}(V)$ as playing the role of a "disc over Y" and $\mathrm{Spec}(K)$ as playing the role of a "punctured disc over Y" (even though for a general valuation ring V its fraction field K is *not* obtained by inverting a single nonzero element of V, so the generic point $\mathrm{Spec}(K)$ in $\mathrm{Spec}(V)$ is *not* open, in contrast with the discretely-valued case).

Grothendieck's "functor of points" criteria are as follows:

Valuative criterion for separatedness for $f : X \to Y$ that is quasi-separated (a mild topological condition that always holds when X is locally noetherian): for every $\mathrm{Spec}(V) \to Y$ and Y-map $h : \mathrm{Spec}(K) \to X$ there is *at most one* extension of h to a Y-map $\mathrm{Spec}(V) \to X$.

In other words, $X_Y(V) \to X_Y(K)$ (composing points in $X_Y(V) = \mathrm{Hom}_Y(\mathrm{Spec}(V), X)$ with $j : \mathrm{Spec}(K) \to \mathrm{Spec}(V)$) is injective ("every rational point extends in at most one way to an integral point"). In terms of a diagram, this says for any commutative square

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{h} & X \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(V) & \longrightarrow & Y \end{array}$$

there is at most one way to fill in the dotted arrow.

Valuative criterion for universal closedness for $f : X \to Y$ that is quasi-compact (e.g., a finite type morphism, or any morphism with X noetherian): for every $\mathrm{Spec}(V) \to Y$ and Y-map $h : \mathrm{Spec}(K) \to X$ there is *at least one* extension of h to a Y-map $\mathrm{Spec}(V) \to X$.

In other words, $X_Y(V) \to X_Y(K)$ (composing points in $X_Y(V) = \mathrm{Hom}_Y(\mathrm{Spec}(V), X)$ with $j : \mathrm{Spec}(K) \to \mathrm{Spec}(V)$) is surjective ("every rational point extends in at least one way to an integral point"). In terms of a diagram, this says for any commutative square

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{h} & X \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(V) & \longrightarrow & Y \end{array}$$

there is at least one way to fill in the dotted arrow.

For a finite type map $f : X \to Y$ that is quasi-separated (e.g., with X locally noetherian, such as a finite type map to a locally noetherian target Y), the synthesis of the valuative criteria of separatedness and universal closedness combine to define:

Valuative criterion for properness for $f : X \to Y$ that is quasi-separated and finite type (e.g., a finite type map with X locally noetherian): for every $\mathrm{Spec}(V) \to Y$ and Y-map $h : \mathrm{Spec}(K) \to X$ there is *exactly one* extension of h to a Y-map $\mathrm{Spec}(V) \to X$.

In other words, $X_Y(V) \to X_Y(K)$ (composing points in $X_Y(V) = \mathrm{Hom}_Y(\mathrm{Spec}(V), X)$ with $j : \mathrm{Spec}(K) \to \mathrm{Spec}(V)$) is bijective ("every rational point extends in exactly one way to an integral point"). In terms of a diagram, this says for any commutative square

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{h} & X \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(V) & \longrightarrow & Y \end{array}$$

there is exactly one way to fill in the dotted arrow.

In this handout we will prove:

Theorem 1.1. *The preceding criteria are necessary and sufficient for the corresponding property* (*separated, universal closed, proper*) *to hold for the class of $f : X \to Y$ in each criterion* (*e.g., quasi-separated morphisms for the property of being separated, quasi-compact morphisms for the property of being universally closed*)*.*

The case of properness immediately reduces to the cases of separatedness and universal closedness due to how properness and the corresponding valuative criterion are defined. In the motivating topological/analytic setting, the criteria with Δ^* and Δ were merely necessary or sufficient, reflecting topological or analytic maps can be much wilder than scheme maps. Loosely speaking, in algebraic geometry there is nothing like "essential singularities" from complex analysis or the topologists' sine curve.

Remark 1.2. The quasi-separated hypothesis in the valuative criterion for separatedness is harmless for two reasons: failure of quasi-separatedness is pathological and quasi-separatedness is a necessary condition for separatedness to hold. Likewise, the hypothesis in the valuative criterion for properness that the map is quasi-separated and finite type consists of necessary conditions for properness to hold.

One might idly wonder the same about the quasi-compactness hypothesis in the valuative criterion for being universally closed: is every universally closed morphism necessarily quasi-compact? Somewhat remarkably, the answer is affirmative! This is irrelevant in practice since one never finds onself trying to prove a map is universally closed without already knowing it is quasi-compact, but (in case you are curious) an affirmative proof due to Bjorn Poonen is given in the answer to a Math Overflow question which asked about exactly this issue.

Remark 1.3. A result of Deligne [EGA IV_4, 18.12.6] (refining an earlier result of Grothendieck [EGA IV_3, 8.11.5]) is that a map of schemes is a closed immersion if and only if it is a proper monomorphism. So by the valuative criterion, this gives a criterion in terms of the functor of points for a finite type map to be a closed immersion (as monicity is a condition on the functor of points and implies separatedness since the diagonal is an isomorphism). There is also a functorial characterization of open immersions [EGA IV_4, 17.9.1]: these are exactly the maps that are étale monomorphisms, where "étale" is a certain important property analogous to "local analytic isomorphism" that can be characterized in terms of the functor of points.

One may then wonder if there is a functorial characterization of immersions, say at least for finite type monomorphusms $f : X \to Y$ between noetherian schemes. The answer is affirmative, but this was missed in [EGA] and is given in a self-contained manner starting on page 100 of S. Mochizuki's book *Foundations of p-adic Teichmüller Theory.* It is a valuative criterion: for any discrete valuation ring V with fraction field K and residue field k, every $h : \text{Spec}(V) \to Y$ that factors through f on the generic and closed points (i.e., the compositions $\text{Spec}(K) \to \text{Spec}(V) \to Y$ and $\text{Spec}(k) \to \text{Spec}(V) \to Y$ factor through f) factors through f. (This is a necessary condition since the only immersion into $\text{Spec}(V)$ through which the generic and closed points factor is $\text{Spec}(V)$ itself – why? – but the proof of sufficiency requires real work.) This result is not common knowledge to algebraic geometers, but it does have some applications.

2. First steps

Let's see that the case of separatedness immediately reduces to the case of universal closedness.

Step 1. Consider $f : X \to Y$ that is quasi-separated (as in the separatedness criterion), so its diagonal $\Delta_f : X \to X \times_Y X$ is quasi-compact (by definition of quasi-separatedness for f). Separatedness for f is equivalent to Δ_f having closed image, in which case the immersion Δ_f is a closed immersion and thus is universally closed (since "closed immersion" is preserved by any base change). Thus, separatedness for the given f is equivalent to universal closedness for its quasi-compact diagonal Δ_f.

Step 2. Next, we have to show for quasi-separated f that the valuative criterion for universal closedness applied to Δ_f expresses *exactly* the valuative criterion for separatedness applied to f. The valuative criterion of separatedness for the quasi-separated f says that for all valuation rings V (with fraction field K) the natural map $X(V) \to Y(V) \times_{Y(K)} X(K)$ is injective. On the other hand, the valuative criterion of universal closedness for the quasi-compact Δ_f says that for all V the natural map

$$X(V) \to (X \times_Y X)(V) \times_{(X \times_Y X)(K)} X(K)$$

is surjective, and the target of this is

$$(X(V) \times_{Y(V)} X(V)) \times_{X(K) \times_{Y(K)} X(K)} X(K) = X(V) \times_{Y(V) \times X(K)} X(V),$$

so the criterion says that any two V-valued points $x, x' \in X(V)$ going to the same place in $Y(V)$ and having the same associated K-valued point in $X(K)$ must be equal. But that is exactly the injectivity from the separatedness criterion for f!

The upshot is that Theorem 1.1 reduces to:

Theorem 2.1. *Let $f : X \to Y$ be a quasi-compact map of schemes. The map f is universally closed if and only if it satisfies the valuative criterion for universal closedness.*

Remark 2.2. In practice, it is useful to know that it is not necessary to check *all* valuation rings in this criterion. For example, when X and Y are locally noetherian and f is finite type, it turns out to be sufficient to check just discrete valuation rings [H, Ch. II, Exer. 4.11] (and with some more commutative algebra, it suffices to use just discrete valuation rings that are complete with algebraically closed residue field).

One can restrict even further in more special situations, as you will learn with experience. For example, when working with finite type schemes over an algebraically closed field k, it is sufficient to consider only $V = k[\![t]\!]$ and k-morphisms $\text{Spec}(V) \to Y$ carrying the closed point of $\text{Spec}\, V$ to a closed point of Y.

We shall now prepare to prove Theorem 2.1, beginning with some simple but useful lemmas.

Lemma 2.3. *Let $i : Z \to X$ be a closed immersion of schemes, and let A be a domain with fraction field K. Let $f : \text{Spec}\, A \to X$ be a map of schemes such that the restriction $f_\eta : \text{Spec}\, K \to X$ of f to the generic point factors* (*necessarily uniquely*) *through i. The map f then factors* (*uniquely*) *through i.*

Proof. Consider the closed subscheme

$$Z' = Z \times_X \text{Spec}(A) \hookrightarrow \text{Spec}\, A$$

obtained by base change of i along f. The hypotheses tells us that this closed subscheme contains the generic point of $\operatorname{Spec} A$, which is to say that the ideal $I \subseteq A$ defining Z' is contained in the prime ideal (0) defining the generic point. That is, $Z' = \operatorname{Spec} A$, and so the projection map $\operatorname{Spec} A = Z' \to Z$ provides the desired factorization. Uniqueness of the factorization is immediate since i is a monomorphism. ■

Lemma 2.4. *If $f : X \to Y$ satisfies the valuative criterion for universal closedness then so does the base change $f' : X' \to Y'$ for any map $Y' \to Y$, and so does $f \circ i$ for any closed immersion $i : Z \hookrightarrow X$.*

Proof. The case of base change is a simple consequence of the universal mapping property of $X' = X \times_Y Y'$ (please check!), and the case of $f \circ i$ follows from Lemma 2.3. ■

Consider the problem of showing that if a quasi-compact map $f : X \to Y$ satisfies the valuative criterion for universal closedness, then f is universally closed. We would like to show that for any map $Y' \to Y$, the base change $f' : X' \to Y'$ is a closed map; that is, $f'(Z')$ is a closed set in Y' for any closed set Z' in X'. By Lemma 2.4, the (quasi-compact!) map f' satisfies the valuative criterion for universal closedness, and if we give Z' its unique structure of reduced closed subscheme of X' then another application of Lemma 2.4 (to the map f' and the closed immersion $i' : Z' \to X'$) shows that $f' \circ i'$ satisfies the valuative criterion for universal closedness. Note also that $f' \circ i'$ is a quasi-compact map (as f' is quasi-compact, and i' is quasi-compact since it is a closed immersion, so their composition is quasi-compact).

Consequently, we may rename $f' \circ i'$ as f and thereby conclude that *to prove the sufficiency* of the valuative criterion for universal closedness it is enough to show that a general quasi-compact $f : X \to Y$ satisfying this valuative criterion has closed image. This completes the "general nonsense", and in the next section we will give the arguments involving genuine theorems with valuation rings that are needed to finish the proof of Theorem 2.1.

3. Necessity of valuative criterion for universal closedness

Let us first show that any scheme morphism $f : X \to Y$ that is universally closed satisfies the valuative criterion for universal closedness. This implication will not make use of the hypothesis that f is quasi-compact, and it will just barely (but in a crucial way at the end) make use of the fact that the valuative criterion involves valuation rings and not arbitrary local domains.

Our aim is to show for any valuation ring V with fraction field K and any given commutative square

$$\begin{array}{ccc} \operatorname{Spec}(K) & \xrightarrow{h} & X \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \operatorname{Spec}(V) & \longrightarrow & Y \end{array}$$

there is at least one way to fill in the dotted arrow. By two applications of Exercise A(i) in HW7, to give such a dotted arrow is the same as to map a section to the base change

$$f' : X_V := X \times_Y \operatorname{Spec}(V) \to \operatorname{Spec}(V)$$

of f and in the given commutative square the map h amounts to exactly a K-valued point of X_V (over $\operatorname{Spec} V$). This base change f' inherits universal closedness from f, so we can replace f with f' to reduce to showing for the case $Y = \operatorname{Spec}(V)$ that a given $x_K \in X(K)$ extends to $X(V)$ (both "points" of X being morphisms over $Y = \operatorname{Spec} V$).

Let Z be the reduced scheme structure on the closure of the image-point of x_K in X, so by closedness of $f : X \to \operatorname{Spec} V$ (due to the universal closedness hypothesis) the image of Z in $Y = \operatorname{Spec} V$ is closed yet (by its definition in terms of x_K) it contains the generic point of $\operatorname{Spec} V$. By construction,the integral scheme Z has function field equal to the residue field K at the image-point of the section x_K of the generic fiber $X_K \to \operatorname{Spec} K$. Since $\operatorname{Spec} V$ is irreducible, the only closed set in $\operatorname{Spec} V$ containing the generic point is the whole space. Thus, there exists $z \in Z$ mapping to the closed point of $\operatorname{Spec} V$.

Now consider the local domain $\mathscr{O}_{Z,z}$ at the point z on the integral scheme Z with function field K. We thereby identify K with the fraction field of the local domain $\mathscr{O}_{Z,z}$, and since f carries z to the closed point of $\operatorname{Spec} V$ we obtain a local map $A \to \mathscr{O}_{Z,z}$ that respects the identification of fraction fields of these domains with K (since the computation of the residue field at the image-point of x_K was via the structure map $X \to \operatorname{Spec} V$).

In other words, $\mathscr{O}_{Z,z}$ is a local domain in the fraction field K of V and it dominates V. Since V is a *valuation ring*, it is maximal with respect to domination in its fraction field! Thus, the map $V \to \mathscr{O}_{Z,z}$ induced by $f : X \to \operatorname{Spec} V$ is an *isomorphism*. In other words, the composite map

$$\operatorname{Spec} \mathscr{O}_{Z,z} \to Z \hookrightarrow X \to \operatorname{Spec} V$$

is an isomorphism, so composing the inverse of this isomorphism with the natural map $\operatorname{Spec} \mathscr{O}_{Z,z} \to X$ gives a V-morphism $x : \operatorname{Spec} V \to X$ (i.e., a section of f) that extends x_K due to how Z was built from x_K. This shows that the original f satisfies the valuative criterion for universal closedness, as desired.

4. Sufficiency of valuative criterion for universal closedness

It remains to treat the most interesting aspect, namely the fact that the valuative criterion for universal closedness actually implies the property of being universally closed, at least for *quasi-compact* maps $f : X \to Y$. As we saw at the end of §2, for this implication it is enough to prove the weaker-sounding claim that a general quasi-compact map $f : X \to Y$ satisfying the valuative criterion for universal closedness has image $f(X)$ that is closed in Y.

Let us first state a useful "specialization" criterion for closedness of the image of a quasi-compact map:

Lemma 4.1. *Let $f : X \to Y$ be a quasi-compact map of schemes. If $f(X)$ is stable under specialization (that is, for each $y \in f(X)$, the closure $\overline{\{y\}}$ is contained in $f(X)$), then $f(X)$ is closed.*

If we were working with noetherian schemes and f were finite type then we could prove the lemma by applying Chevalley's constructibility theorem [H, Ch. II, Exer. 3.19] to get the constructibility of $f(X)$ in Y and then apply the specialization criterion for closedness of constructible sets [H, Ch. II, Exer. 3.18] to confirm that $f(X)$ is closed in Y. But this would be insufficient for our needs since this Lemma is really being applied after an arbitrary base change on the original map in the valuative criterion and so it requires us to allow non-noetherian Y (such as the spectrum of a valuation ring). Regardless of that insufficiency, the real thrust of Lemma 4.1 is that it works without noetherian or finite-type hypotheses.

Proof. If $\{U_i\}$ is an open covering of Y, with $f_i : f^{-1}(U_i) \to U_i$ the restriction of f over U_i, then $f(X) \cap U_i$ is the image of f_i, and so $f(X)$ is closed in Y if and only if f_i has closed image in U_i for all i. Since each f_i has image in U_i that is stable under specialization in U_i (because formation of closure is compatible with intersecting with an open subset), and each f_i is quasi-compact, it is therefore enough to prove the result for each of the f_i's; that is, we can work locally on Y. Thus, we can assume $Y = \operatorname{Spec} R$ is affine, so X is quasi-compact. If $\{V_j\}$ is a finite open affine cover of X then we can replace X with the disjoint union $\coprod V_j$ that has the same image in Y to reduce to the case where X is also affine, say $X = \operatorname{Spec} B$.

We may also replace R with its quotient by the kernel of $R \to B$ (as this replaces Y with a closed subset containing $f(X)$) to reduce to the case when $R \to B$ is injective. In this case we will prove that the subset $f(X) \subseteq Y$ contains all generic points of Y and thus by its assumed stability under specialization is equal to Y (as every point of a scheme is a specialization of a generic point, since every point lies in an irreducible component; alternatively, in the affine setting as at present we are using that every prime ideal of a ring contains a minimal prime).

Let η be a minimal prime of R, so by localizing we have an injection $R_\eta \to B_\eta$. By injectivity of $R_\eta \to B_\eta$ and the non-vanishing of R_η, the ring B_η is nonzero. Thus, B contains a prime $\mathfrak{q}$ whose contraction to R is contained in η and thus (by minimality of η) is equal to η. This shows that $f(X)$ contains all generic points of Y, as we saw is sufficient for our needs. ■

Returning to our setup, we wish to prove that if $f : X \to Y$ is quasi-compact and satisfies the valuative criterion for universal closedness then the subset $f(X) \subset Y$ is stable under specialization. Let $y = f(x)$ for a point $x \in X$, and let $y_0 \in Y$ be a specialization of y. We wish to construct a point $x_0 \in X$ such that $f(x_0) = y_0$. Here is the key ingredient where valuation rings play an essential role:

Lemma 4.2. *If S is a scheme and $s, t \in S$ are two points then s is a specialization of t if and only if there exists a valuation ring A and a map $g : \mathrm{Spec}(A) \to S$ carrying the closed point to s and generic point to t. For locally noetherian S and $s \neq t$, the same holds limiting A to be a discrete valuation ring.*

In effect, this says that maps from valuation rings are sufficient to probe the specialization relation among points in a scheme (and likewise with discrete valuation rings when S is locally noetherian).

Proof. If such a g exists then the preimage $g^{-1}(\overline{\{t\}})$ is a closed subset of the irreducible $\mathrm{Spec}(A)$ that contains the generic point and so is the entire space. Thus, this preimage contains the closed point and hence the image s of the closed point lies in the closure of t, as desired.

Going in reverse, if s is a specialization of t then any affine open $\mathrm{Spec}(R)$ around s must also contain t (otherwise the closed set $S - \mathrm{Spec}(R)$ would contain t and hence also its closure, contradicting that the specialization s of t lies in $\mathrm{Spec}(R)$ by design; in general what is going on is that specializations of a point in a scheme correspond to prime ideals of the local ring of the point, which is computed using any affine open around the point). Hence, for the purposes of constructing the desired g we can replace S with $\mathrm{Spec}(R)$.

Now our assertion is that if there is a containment of primes $\mathfrak{p} \subset \mathfrak{q}$ in a ring R then there is a map $R \to A$ to a valuation ring A such that its kernel (i.e., the contraction of (0)) is $\mathfrak{p}$ and the contraction of the maximal ideal of A is $\mathfrak{q}$. In other words, we seek a local injection of domains $D := (R/\mathfrak{p})_{\mathfrak{q}} \to A$. Even better, we can find such an A inside $\mathrm{Frac}(D)$ since every local subring of a field is dominated by a maximal one and the maximal ones are precisely the valuation rings with that field as its fraction field.

In case S is locally noetherian and $s \neq t$, the local domain D as built above is noetherian and not a field. Hence, for the more refined assertion with dscrete valuation rings in such cases we just have to check that any noetherian local domain D that is not a field is dominated by a discrete valuation ring inside $\mathrm{Frac}(D)$. This is a special case of [H, Ch. II, Exer. 4.11(a)] (the case $L = K$ in the notation used there), which relies on the Krull-Akizuki Theorem [Mat, Thm. 11.7]. ∎

By Lemma 4.2, there exists a valuation ring V and a map $h : \mathrm{Spec}\, V \to Y$ such that h carries the generic point to $y = f(x)$ and the closed point to y_0. Using Lemma 2.4, we may replace $f : X \to Y$ with its base change by h to reduce to the case $Y = \mathrm{Spec}\, V$ with a point x in the fiber X_K of f over the generic point $\mathrm{Spec}\, K$ of Y, and we seek to find $x_0 \in X$ over the closed point y_0 of $Y = \mathrm{Spec}\, V$. (Indeed, making such a point x_0 on the scheme that was $X \times_Y \mathrm{Spec}\, V$ before renaming yields the point we really want on the original X by passing to the image point under $\mathrm{pr}_1 : X \times_Y \mathrm{Spec}\, V \to X$.)

We will build the desired x_0 using an *another* valuation ring V', one whose fraction field is $k(x)$, by applying the valuative criterion for universal closedness with this auxiliary valuation ring. In effect, the role of V here is two-fold: to probe specialization relations in the image of the original f and as an intermediate device in the construction of V' to which the valuative criterion will be applied.

By design, the residue field $k(x)$ is an extension of the residue field K of the generic point y of $Y = \mathrm{Spec}\, V$. Now the miracle happens: the valuation ring V of K is dominated by a valuation ring V' of the extension field $k(x)$, since every local subring of a field (such as $V \subset k(x)$) is dominated by a valuation ring with fraction field equal to the given field. (Another way to say this is that valuations may always be extended through field extensions, at the cost of possibly enlarging the value group.) The natural map

$$\xi : \mathrm{Spec}\, V' \to \mathrm{Spec}\, V = Y$$

sends the closed point to y_0 (since $V \to V'$ is local) and has generic-point restriction $\mathrm{Spec}\, k(x) \to \mathrm{Spec}\, V$ that factors through $f : X \to \mathrm{Spec}\, V$ via the natural map $\mathrm{Spec}\, k(x) \to X$. Thus, *by the valuative criterion for universal closedness* (which holds for f by hypothesis!) the map $\xi : \mathrm{Spec}\, V' \to \mathrm{Spec}\, V$ factors through $f : X \to \mathrm{Spec}\, V$, and the resulting map $\mathrm{Spec}\, V' \to X$ must (by compatibility with ξ) carrying the closed point of $\mathrm{Spec}\, V'$ to a point $x_0 \in X$ that lies over the closed point y_0 of $Y = \mathrm{Spec}\, V$! This completes the verification of the specialization criterion for closedness of $f(X)$ (using f and X as in the initial setup before we replaced the original base Y with $\mathrm{Spec}\, V$).