“An idea which can be used only once is a trick. If one can use it more than once, it becomes a method.” Pólya/Szegö

Read section 25 of Matsumura (it is self-contained). Do Exercise 25.4, showing your isomorphisms to be compatible with the natural d-maps. Moreover, if $A$ and $B$ are two $R$-algebras, construct a natural isomorphisms of $A \otimes_R B$-modules

$$\Omega^1_{A \otimes_R B/R} \simeq \Omega^1_{A \otimes_R B/A} \oplus \Omega^1_{A \otimes_R B/B}$$

compatibly with the d-maps.

Read §6 in Chapter 2 and do the following exercises that work out carefully the basic theory of projective bundles (which Hartshorne will introduce in §7 of Chapter 2 in a manner that is more ad hoc). In particular, this conceptually explains things like the Segre embedding and the “points” of projective space.

1. Let $X$ be a locally ringed space, $\mathcal{E}$ an $\mathcal{O}_X$-module. We are interested in understanding invertible quotients of $\mathcal{E}$; that is, surjections $\mathcal{E} \to \mathcal{L}$ with $\mathcal{L}$ locally free of rank 1. We say that two invertible quotients $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{E}$ are isomorphic if there exists an isomorphism $\varphi : \mathcal{L}_1 \simeq \mathcal{L}_2$ compatible with the maps from $\mathcal{E}$ (in which case $\varphi$ is unique and this relation is clearly symmetric and transitive). Explain why there is a natural identification between ‘isomorphism classes’ of invertible quotients of $\mathcal{E}$ and a certain set of $\mathcal{O}_X$-submodules of $\mathcal{E}$ (so ‘the’ quotients by these are distinguished isomorphism class representatives).

2. Consider the scheme $\mathbf{P} = \text{Proj}(\mathbf{Z}[T_0,\ldots,T_n])$ with the invertible sheaf $\mathcal{O}_\mathbf{P}(1)$. This sheaf has $n+1$ global sections $T_0,\ldots,T_n$, with $\pi : \mathcal{O}_\mathbf{P}^{\oplus(n+1)} \to \mathcal{O}_\mathbf{P}(1)$ surjective. Prove that this is the ‘universal’ scheme equipped with an invertible quotient of a globally free rank $n+1$ sheaf. More precisely, let $X$ be a scheme, $p : \mathcal{O}_X^{\oplus(n+1)} \to \mathcal{L}$ a surjection onto an invertible sheaf $\mathcal{L}$. Prove that there is a unique map $f : X \to \mathbf{P}$ such that the surjection map

$$\mathcal{O}_X^{\oplus(n+1)} \simeq f^*(\mathcal{O}_\mathbf{P}^{\oplus(n+1)}) \xrightarrow{f^*(\pi)} f^*(\mathcal{O}_\mathbf{P}(1))$$

is isomorphic to $p$. From this point of view, what is the ‘meaning’ of the part of $X$ that maps into $D_+(T_i)$?

(optional) Prove an analogue in the context of analytic spaces, using analytic $\mathbf{P}_\mathbf{C}^n$ and some invertible sheaf.

3. Using the technique of base change from $\mathbf{Z}$, if $A$ is any ring, explain why $\mathbf{P}^n_A$ and the natural surjection $\mathcal{O}_{\mathbf{P}^n_A}^{\oplus(n+1)} \to \mathcal{O}_{\mathbf{P}_A^n}(1)$ are suitably ‘universal’ for $A$-schemes. Deduce that $\mathbf{P}^n_A(A)$ is identified with the set of isomorphism classes of $A$-modules $M$ with $n+1$ specified generators, and for which $\widetilde{M}$ is locally free of rank 1. For a local ring $A$, recover the description of $\mathbf{P}^n_A(A)$ found in class last term; what if $A$ is a Dedekind domain? Explain why $\mathbf{P}^n_k(k)$ is ‘dual’ to the classical point of view.

We want to generalize the above to sheaves which might not be globally free. This is going to be done via Yoneda-type arguments. When you are done, convince yourself that it could have been done without Yoneda, but that the Yoneda Lemma lets you express the essential ideas much more effectively.
We now work within the category of schemes (or complex-analytic spaces if you like that). Let $X$ be an object, $\mathcal{E}$ an $\mathcal{O}_X$-module. If $p : T \to X$ is a morphism, we write $\mathcal{E}_T$ for $p^*(\mathcal{E})$.

4. For any morphism $p : T \to X$, let $I_X(T)$ denote the set of invertible quotients of $\mathcal{E}_T$ on $T$. Explain why this behaves functorially in $T$, and why for an open covering $\{X_i\}$ of $X$, with $\{T_i\}$ the induced open covering of $T$, there is a natural exact sequence of sets

$$I_X(T) \to \prod I_{X_i}(T_i) \Rightarrow \prod I_{X_i}(T_i \cap T_j).$$

Here, the second product is indexed by unordered pairs $\{i, j\}$ of indices and $I_{X_i}$ denotes $I_{X_i, \mathcal{E}|X_i}$, so $I_{X_i}(T_i)$ is naturally identified with $I_X(T_i)$. Thus, the functor $I_X$ is like a ‘Zariski sheaf’ of sets.

*5. Suppose that there exists some $p : P \to X$ which represents the functor $I_X$. Explain why Yoneda asserts that the specification of a natural isomorphism $I_X \cong \text{Hom}_X(\cdot, P)$ is equivalent to the specification of an invertible quotient $\pi : \mathcal{E}_P \to \mathcal{L}$ on $P$ which is ‘universal’: that is, for any $g : T \to X$ and invertible quotient $\mathcal{E}_T \to \mathcal{L}'$, there is a unique $X$-map $f : T \to P$ so that the invertible quotient $\mathcal{E}_T \cong (p \circ f)^*(\mathcal{E}) \cong f^*(\mathcal{E}_P) \to f^*(\mathcal{L})$ (using $f^*(\pi)$ for the second map) is isomorphic to $\mathcal{L}'$ (via a necessarily unique isomorphism).

In particular, if $U$ is open in $X$ then show that $p_U : P^{-1}(U) \to U$ and $\mathcal{L}|p^{-1}(U)$ (viewed as a quotient of $(p_U)^*(\mathcal{E}|U) \cong \mathcal{E}_P|p^{-1}(U)$) is a ‘universal pair’ representing the functor $I_{U, \mathcal{E}|U}$ on objects over $U$.

*6. In the above notation, note that $I_X(T_i \cap T_j)$ is naturally bijective with $I_{X_i \cap X_j}(T_i \cap T_j)$. Use this to show that if $I_{X_i}$ is representable by some locally ringed space $p_i : P(\mathcal{E}|X_i) \to X_i$ (so we are given the data $I_{X_i} \cong \text{Hom}_{X_i}(\cdot, P(\mathcal{E}|X_i))$ as functors on objects over $X_i$ too), then the functors determine $X_i \cap X_j$-maps

$$p_i^{-1}(X_i \cap X_j) \cong p_j^{-1}(X_i \cap X_j)$$

which satisfy the gluing lemma, so we can glue to get an object $P(\mathcal{E})$ and a morphism $p : P(\mathcal{E}) \to X$, together with a functorial isomorphism $\text{Hom}_X(\cdot, P(\mathcal{E})) \cong I_{X, \mathcal{E}}$. In other words, representability of $I_X$ is ‘local on $X$’.

On each $P(\mathcal{E}|X_i)$ there is a ‘universal’ invertible sheaf quotient $\mathcal{L}_i$ of $p_i^*(\mathcal{E}|X_i)$; why do these glue to give an invertible quotient $p^*(\mathcal{E}) \to \mathcal{L}$ which is ‘universal’?

*7. Show that if $I_{X, \mathcal{E}}$ is representable when $\mathcal{E} = \mathcal{O}_X^{\oplus(n+1)}$ and $X$, $n \geq 0$ are arbitrary, prove that $I_{X, \mathcal{E}}$ is always representable for $\mathcal{E}$ of finite presentation (use certain closed subspaces, paying attention to finite presentation). Also, show that if $I_{X, \mathcal{E}}$ is representable by some ‘universal pair’ $p : P(\mathcal{E}) \to X$ and $\pi : p^*(\mathcal{E}) \to \mathcal{L}$, and if $f : X' \to X$ is a morphism, then $I_{X', f^*(\mathcal{E})}$ is representable; more explicitly, use Yoneda to see that $q_2 : P(\mathcal{E}) \times_X X' \to X'$ and

$$(q_2)^*(f^*(\mathcal{E})) \cong (f \circ q_2)^*(\mathcal{E}) = (p \circ q_1)^*(\mathcal{E}) \cong (q_1)^*(p^*(\mathcal{E})) \xrightarrow{(q_1)^*(\pi)} (q_1)^*(\mathcal{L})$$

is a ‘universal pair’ (this generalizes the end of Exercise 5).

8. Let $Z$ be a final object in our category. Show that the representability of $I_{Z, \mathcal{O}_Z^{\oplus(n+1)}}$ for all $n \geq 0$ implies representability of $I_{X, \mathcal{E}}$ for all $X$ and all finitely presented $\mathcal{E}$ on $X$. 


9. Recall that the functor “global sections of $\mathcal{O}$” is represented by an object denoted $A^1_Z$ (so this also specifies a distinguished global section $\xi$); let $U \subset A^1_Z$ be the open subspace on which $\xi$ is non-vanishing. Show $I_{Z,\mathcal{O}^\oplus(n+1)}$ is represented by a gluing of copies $D_j$ of $A^n_Z$ ($0 \leq j \leq n$) with the “factors” $A^1_Z$ for $D_j$ indexed by $\{0, \ldots, n\} - \{j\}$, where for $j' \neq j$ we glue $D_{j'}$ to $D_j$ by using $\xi \mapsto 1/\xi$ to identify the copies of $U$ in the respective $A^1_Z$-factors indexed by $j$ and $j'$.

*10. Conclude from Exercise 9 (without reference to Exercise 2!) that $I_{X,\mathcal{E}}$ is representable for $\mathcal{E}$ of finite presentation. One usually writes $P(\mathcal{E})$ and $\mathcal{O}_{P(\mathcal{E})}(1)$ (or just $\mathcal{O}(1)$) for an associated ‘universal pair’ (this implicitly includes a surjection $\pi_\mathcal{E}$ from the pullback $\mathcal{E}_{P(\mathcal{E})}$ onto $\mathcal{O}_{P(\mathcal{E})}(1)$).

Let’s make this explicit (and more general) for schemes. When $X$ is affine and $\mathcal{E}$ is any quasi-coherent sheaf, let $S(\mathcal{E})$ denote the symmetric algebra of $M = \Gamma(X, \mathcal{E})$ over $\Gamma(X, \mathcal{O}_X)$ (when $\mathcal{E} = \mathcal{O}_X^{\oplus(n+1)}$, what is this?). Using adjointness, define a map $p^*(\mathcal{E}) \to S(\mathcal{E})(1)$ on the $X$-scheme $p : \text{Proj}(S(\mathcal{E})) \to X$. Working locally on $\text{Proj}(S(\mathcal{E}))$, check this sheaf map is surjective with invertible target. Explain why this is a universal pair for the functor $I_{X,\mathcal{E}}$ (i.e., it makes explicit the above abstract nonsense), and describe the fibers of $p$. Globalize as follows: if $X$ is a scheme and $\mathcal{E}$ is a quasi-coherent sheaf, define a graded quasi-coherent ‘symmetric’ $\mathcal{O}_X$-algebra $S(\mathcal{E})$. For any $\mathbb{N}$-graded quasi-coherent $\mathcal{O}_X$-algebra $S$, define an $X$-scheme $\text{Proj}(S)$ in a manner analogous to $\text{Spec}$, but using ‘Proj’ in place of ‘Spec,’ and then show that $p : \text{Proj}(S(\mathcal{E})) \to X$, together with a natural invertible quotient of $p^*(\mathcal{E})$, represents the functor $I_{X,\mathcal{E}}$.

*11. Pick an object $X$ and two finitely presented sheaves $\mathcal{E}$ and $\mathcal{F}$ on $X$ (or any two quasi-coherent sheaves if $X$ is a scheme). Describe a natural map of functors $I_{X,\mathcal{E}} \times I_{X,\mathcal{F}} \to I_{X,\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{F}$ (natural in $\mathcal{E}$ and $\mathcal{F}$ too!), and prove it is a monomorphism (or ‘subfunctor’). For schemes or analytic spaces, this yields a natural map $P(\mathcal{E}) \times_X P(\mathcal{F}) \to P(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$, automatically ‘associative’, compatible with base change, etc. (why?). Prove it is a closed immersion (hint: reduce to the case where $\mathcal{E}$ and $\mathcal{F}$ are globally free and then compute). If $\mathcal{E} = \mathcal{O}_X^{\oplus(n+1)}$, $\mathcal{F} = \mathcal{O}_X^{\oplus(m+1)}$, so $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \simeq \mathcal{O}_X^{\oplus(n+1)(m+1)}$, check that the resulting map is nothing other than the Segre map!! How about a ‘globalized’ $d$-uple embedding?

*12. Let $X$ be a scheme (resp. analytic space) and $\mathcal{E}$ a quasi-coherent sheaf (resp. a finitely presented sheaf), and let $\pi_\mathcal{E} : \mathcal{E}_P \to \mathcal{O}_P(1)$ the universal invertible sheaf quotient of $\mathcal{E}_P$. For an invertible sheaf $\mathcal{L}$ on $X$, define $\mathcal{E}_\mathcal{L} = \mathcal{P}(\mathcal{E} \otimes \mathcal{L})$. Explain why there is a unique $X$-isomorphism $f : \mathcal{P}_\mathcal{L} \simeq \mathcal{P}$ for which $(\mathcal{E} \otimes \mathcal{L})_{\mathcal{P}_\mathcal{L}} \simeq f^*(\mathcal{E}_P) \to f^*(\mathcal{O}_P(1))$ is isomorphic to the quotient map $\pi_{\mathcal{E} \otimes \mathcal{L}}$ (loosely, such that $f$ pulls $\mathcal{O}_P(1)$ back to $\mathcal{O}_{\mathcal{P}_\mathcal{L}}(1)$). What is this saying about $\mathcal{P}_A^n(A)$ for a local ring $A$?