Math 216B Homework 8

“. . . computational procedures have become so complicated that any progress by those means has become impossible, without the elegance which modern mathematicians have brought to bear on their research, and by means of which the spirit comprehends quickly and in one step a great many computations.”

Evariste Galois

In Matsumura, read §17 to learn about Cohen-Macaulay rings and modules (which will play an important role in Serre duality), and read Theorem 23.1 and its Corollary.

Note that even if you only wanted these latter results for regular rings, the proofs do not work unless you proceed more generally with Cohen-Macaulay rings (since the CM property behaves well with respect to regular sequence constructions, whereas the regular property does not). A nice immediate consequence of these Matsumura results is that if $f : X \to Y$ is a finite map between regular schemes of finite type over a field, with $X$ and $Y$ equidimensional of dimension $d$ (i.e., all irreducible components have dimension $d$), then $f$ is automatically flat! A similar result is true over certain bases more general than a field, but one needs to be a bit careful about how (non-local) dimension theory and closed points behave.

Read §5 in Ch. III of Hartshorne. Do Exercises 3.1, 3.2, *3.3 (make (b) and (c) be $\delta$-functorial), *3.4, *3.6, 3.7 (extend (a) to a $\delta$-functorial comparison between higher cohomology modules and Ext modules), 4.2, *4.4, 4.5, *4.10, *4.11. Note that the isomorphism in 4.5 (which goes from Pic to cohomology) implicitly involves a choice of sign, so you must fix this choice for the rest of your life in order to avoid confusion.

Extra 1: Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of ringed spaces. Let $\mathcal{U} = \{U_i\}$ be an open covering of $Y$ and $\mathcal{V} = \{V_j\}$ an open covering of $X$, with $V_j \subseteq f^{-1}(U_{\tau(j)})$ for some $\tau : J \to I$ (i.e., $\mathcal{V}$ is a refinement of $f^{-1}(\mathcal{U}) = \{f^{-1}(U_i)\}$). Define $H^n(\mathcal{U}, \mathcal{F}) \to H^n(\mathcal{V}, f^*(\mathcal{F}))$, naturally in $\mathcal{F}$ (and independent of the choice of $\tau$), in a manner which is suitably compatible with the refinement maps in Čech cohomology on each side. Discuss what happens for composite maps of ringed spaces.

Prove moreover that for any $\mathcal{O}_Y$-module $\mathcal{F}$ and any integer $i$ and any $\mathcal{O}_X$-module map $f^*(\mathcal{F}) \to \mathcal{G}$, the diagram

$$
\begin{array}{ccc}
H^i(\mathcal{U}, \mathcal{F}) & \to & H^i(\mathcal{V}, \mathcal{G}) \\
\downarrow & & \downarrow \\
H^i(Y, \mathcal{F}) & \to & H^i(X, \mathcal{G})
\end{array}
$$

commutes. Passing to direct limits over covers gives a commutative diagram with the ‘true’ Čech cohomology. This diagram provides the means by which one begins to study the behavior of cohomology with respect to base change (generalizing the study of pushfoward commuting with flat base change).

Extra 2: (i) Prove that blow-ups naturally commute with flat base change, via maps defined by universal properties.

(ii) Let $V = V(\mathcal{O}(-1))$ be the canonical line bundle over $\mathbb{P}^n_R$. Define an $R$-morphism $V \to \mathbb{A}^{n+1}_R$ which is an isomorphism away from the zero-section and explicitly realize this map as the blow-up along the zero-section. Keep in mind the transition functions involved over affine overlaps.