Math 216B Homework 9

“A theorem is not true anymore because one can draw a picture; it is true because it is functorial.” Serge Lang

Read §19 up through 19.4 in Matsumura, §6–§7 in Ch. III Hartshorne (6.11A, 6.12A use §19 of Matsumura).

In Ch. III of Hartshorne, do Exercises 5.1 (with $X$ proper over an artinian ring $A$ and dimension replaced by $A$-length), *5.2 (in (a) consider $X$ proper over an artinian ring $A$, replacing dimension with $A$-length and field $k$ and replacing $\mathcal{O}_X(1)$ with any ample invertible sheaf; first treat the case $A$ is a field and then reduce to that case), 5.3(a),(b), *5.7, 5.8 (extra credit to deduce it any field), *5.9 (show that in terms of Čech cohomology, $\delta(\mathcal{O}(1)) = (uv)^{-1}dv \wedge du$ where $u = x_1/x_0$ and $v = x_2/x_0$, and prove the assertion in characteristic $p > 0$ at the end of the Note by using that coherent cohomology is $p$-torsion), *6.1.

*Extra 1: Read §22 in Matsumura. Using this, prove the following important theorem, which often reduces flatness questions in a relative setting to a ‘classical’ setting of geometric fibers. Let $S$ be a locally noetherian scheme, $f : X \rightarrow Y$ an $S$-morphism between locally noetherian $S$-schemes, and $\mathcal{F}$ an $S$-flat coherent $\mathcal{O}_X$-module. Assume also that $Y$ is $S$-flat. If $\mathcal{F}_s$ on $X_s$ is $f_s$-flat for all $s \in S$ then prove that $\mathcal{F}$ is $f$-flat (the converse is trivial). In particular, if $X$ and $Y$ are $S$-flat then $f$ is flat if and only if every induced map $f_s$ between (geometric) fibers over $S$ is flat.

Read §24 in Matsumura up to Theorem 24.3, yielding the Theorem on Generic Flatness: if $f : X \rightarrow Y$ is a map of finite type with $Y$ noetherian and integral then for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists a non-empty open set $U$ in $Y$ such that $\mathcal{F}|_{f^{-1}(U)}$ is $U$-flat.

*Extra 2 (preparation for Serre duality): Let $(X, \mathcal{O}_X)$ be a ringed space.

(i) Consider the ‘bi-derived’ functors (in sense of HW7, Extra 4) $\text{Ext}_X^\bullet(\cdot, \cdot)$ and $\mathcal{E}xt_X^\bullet(\cdot, \cdot)$ of $\text{Hom}_{\mathcal{O}_X}(\cdot, \cdot)$ and $\mathcal{H}om_{\mathcal{O}_X}(\cdot, \cdot)$ respectively. Let $U \subseteq X$ be open. Consider the bifunctor from $\mathcal{O}_X$-mod to $\Gamma(U, \mathcal{O}_X)$-mod given by $\mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$. Define $\mathcal{E}xt_U^\bullet(\mathcal{F}, \mathcal{G})$ to be its derived bifunctor. Prove the existence of a unique isomorphism between this and the bi-δ-functor $\mathcal{E}xt^\bullet_U(\mathcal{F}|_U, \mathcal{G}|_U)$ (composing restriction to $U$ with the $\mathcal{E}xt^\bullet_U$ bifunctor on $\mathcal{O}_U$-modules), extending the evident isomorphism in degree 0. Carry out an analogue for $\mathcal{E}xt$.

(ii) For open sets $U \subseteq V$ in $X$, define a unique restriction map $\mathcal{E}xt^\bullet_V \rightarrow \mathcal{E}xt^\bullet_U$ of bi-δ-functors, respecting module structures via the ring map $\Gamma(V, \mathcal{O}_X) \rightarrow \Gamma(U, \mathcal{O}_X)$ and coinciding with the evident restriction map in degree 0. Check that this is bifunctorial with respect to composite inclusions of opens. In this way, we get a bi-δ-functor $\mathcal{O}_X$-presheaf, and we can sheafify this. Construct a canonical identification between this and $\mathcal{E}xt^\bullet_X$ as bi-δ-functorial $\mathcal{O}_X$-modules.

(iii) If $X$ is a locally noetherian scheme and $\mathcal{F}$, $\mathcal{G}$ are coherent, prove $\mathcal{E}xt^\bullet_X(\mathcal{F}, \mathcal{G})$ is coherent for every $n$. Be explicit in the affine case (i.e., bi-δ-functorially identify the global sections with ordinary module $\text{Ext}$’s); Exer. 7.7 in Matsumura (with $B = A_f$) may be helpful here.

(iv) Construct in general a bi-δ-functorial map of $\mathcal{O}_{X,x}$-modules

$$\mathcal{E}xt^\bullet_X(\mathcal{F}, \mathcal{G})_x \rightarrow \text{Ext}^\bullet_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

extending the evident map in degree 0. Prove that if $X$ is a locally noetherian scheme and $\mathcal{F}$ and $\mathcal{G}$ are coherent, then these maps are isomorphisms.