Math 216C. The Theorem on Formal Functions

In Hartshorne, Ch. III, section 11, he proves a version of the Theorem on Formal Functions (omitting number of compatibilities/functorialities implicit in the definitions that are used quite frequently in the proofs). The result is only given in the case of completion with respect to a maximal ideal and only the projective case is treated, essentially via an explicit calculation for $\mathcal{O}(n)$’s (and then using standard arguments with these sheaves to treat the general projective case). Focusing on the projective case is likely because the book only proves coherence of cohomology in that case.

Nevertheless, the (clever) computational use of projective methods in the proof of the Theorem on Formal Functions can be replaced by a completely different conceptual method of proof in EGA (due to Serre) which treats the general proper case (without proceeding via Chow’s Lemma to reduce to the projective case) and which handles completion with respect to any ideal:

**Theorem.** Let $X \to \text{Spec}(A)$ be proper, with $A$ noetherian, and $\mathcal{F}$ coherent on $X$. Let $I$ be an ideal in $A$, $\hat{A}$ the $I$-adic completion of $A$. Let $X_n = X \times_A (A/I^n + 1)$ (the $n$th infinitesimal neighborhood of the fiber over $I = 0$) and let $\mathcal{F}_n$ be the pullback of $\mathcal{F}$ to $X_n$.

For each integer $i$, the natural ‘pullback’ map of $\hat{A}$-modules

$$\hat{A} \otimes_A H^i(X, \mathcal{F}) \to \lim_{\leftarrow} H^i(X_n, \mathcal{F}_n)$$

is an isomorphism (the cohomologies on $X_n$ are all computed on the same space $X_0$).

Moreover, using the inverse limit topology on the right side, with each $H^i(X_n, \mathcal{F}_n)$ discrete, and using the natural $I$-adic topology on the finite $\hat{A}$-module on the left side, the above isomorphism is a topological one.

This is a vast generalization (to the proper case, in place of the finite case) of the fact that the $I$-adic completion of a finite module over a noetherian ring $A$ can be computed by base change to the $I$-adic completion of $A$. It should also be noted that we do not claim that the transition maps in the inverse limit are surjective for large $n$. Before giving the proof, we make some auxiliary remarks.

First, since $A \to \hat{A}$ is flat, the left side is naturally identified with $H^i(X \otimes_A \hat{A}, \mathcal{F} \otimes_A \hat{A})$ (i.e., cohomology computes after passing to the base change by $A \to \hat{A}$). In this way, it follows from functoriality of pullback maps in sheaf cohomology that it suffices to prove the theorem in the case where $A$ is $I$-adically separated and complete. We won’t actually reduce to this special case, but it should be emphasized that this is a very natural context in which to contemplate the result. The reason is that over such $A$ (or rather, over the formal scheme $\text{Spf}(A)$), one can also consider an associated (proper) formal scheme $\hat{X}$ and an associated (coherent) sheaf $\hat{\mathcal{F}}$ and naturally identify both sides of the Theorem on Formal Functions with $H^i(\hat{X}, \hat{\mathcal{F}})$.

This gives a GAGA-style theorem identifying the cohomology of a coherent sheaf $\mathcal{F}$ on a proper scheme $X$ over $\text{Spec}(A)$ and the cohomology of a certain (coherent) sheaf $\hat{\mathcal{F}}$ on a certain (proper) formal scheme $\hat{X}$ over $\text{Spf}(A)$ (passing from schemes over $\text{Spec}(A)$ to formal schemes over $\text{Spf}(A)$ is analogous to analytification over $C$). This only makes sense (for the $I$-adic topology on $A$) when $A$ is $I$-adically separated and complete (and noetherian,
so coherence is a good notion). As in the analytic case over C, one can get deeper results concerning “algebraicity” of maps and sheaves in the setting of certain formal schemes (called the Grothendieck Existence Theorem).

The second point is that it is possible to “globalize” the result in terms of higher direct image sheaves, but we won’t do this (it just amounts to developing the right terminology, and the hard work ultimately comes down to the case we’ll treat). Nevertheless, one useful version of this Theorem in terms of higher direct image sheaves is the following. Let \( f : X \to Y \) be a proper map, with \( Y \) locally noetherian, \( y \in Y \), \( \mathcal{F} \) coherent on \( X \). Then the natural map of \( \mathcal{O}_y \)-modules

\[
(R^i f_*(\mathcal{F}))_y \to \lim_{\mathcal{I}} H^i(X_y, \mathcal{F}_{y,n})
\]

is an isomorphism, where \( \mathcal{F}_{y,n} \) is the pullback of \( \mathcal{F} \) to \( X \times \text{Spec}(\mathcal{O}_y / \mathfrak{m}_y^n) \) (which has underlying space \( X_y = f^{-1}(y) \)) and the left side denotes the \( \mathfrak{m}_y \)-adic completion of the finite \( \mathcal{O}_y \)-module \( R^i f_*(\mathcal{F})_y \). This result (which is also true in the setting of analytic spaces, by a completely different proof) is a first step towards passing from infinitesimal information along a fiber to properties about the more globalized higher direct image sheaves near the fiber, and thereby (hopefully) to results about cohomology along nearby fibers.

This is particularly effective for proving vanishing theorems globally by proving vanishing theorems along fibers. The theory of cohomology and base change (discussed in Ch. III, section 12 and more elegantly in Mumford’s Abelian Varieties, Ch. 2, section 5) provides the necessary technical results to carry out this sort of idea, provided one assumes that \( \mathcal{F} \) is \( Y \)-flat. Now we come to the proof of the theorem.

**Proof.** Note that all cohomology in the theorem can be computed on the space \( X \), so we do this in the proof (i.e., we view \( \mathcal{F}_n \) as a coherent sheaf on \( X \) which happens to be killed by \( I^n \)). We write \( I^n \mathcal{F} \) for the evident coherent subsheaf of \( \mathcal{F} \) on \( X \) (namely, if \( \mathcal{F} \) is the ‘pullback’ of \( \mathcal{I} \) to a coherent ideal sheaf on \( X \), then \( I^n \mathcal{F} \) is the image of \( I^n \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F} \)). From the exact sequence \( 0 \to I^n \mathcal{F} \to \mathcal{F} \to \mathcal{F}/I^n \mathcal{F} \to 0 \), we get a slightly long exact sequence

\[
H^i(I^n \mathcal{F}) \xrightarrow{\text{pullback}} H^i(\mathcal{F}) \to H^i(\mathcal{F}/I^n \mathcal{F}) \to H^{i+1}(I^n \mathcal{F}) \xrightarrow{\text{pullback}} H^{i+1}(\mathcal{F}).
\]

Here and below, all cohomology will be computed on \( X \) unless otherwise specified, so we have suppressed the \( X \) in the cohomology notation. Observe that the natural map \( H^\bullet(\mathcal{F}) \to H^\bullet(\mathcal{F}/I^n \mathcal{F}) \) is identified with the pullback map \( H^\bullet(X, \mathcal{F}) \to H^\bullet(X_n, \mathcal{F}_n) \), so the maps \( H^i(\mathcal{F}) \to H^i(\mathcal{F}/I^n \mathcal{F}) \) and \( H^i(\mathcal{F}/I^{n+1} \mathcal{F}) \to H^i(\mathcal{F}/I^n \mathcal{F}) \) are the ‘same’ as the ones in the Theorem. More precisely, if we consider the induced short exact sequence of \( A \)-modules

\[
0 \to H^i(\mathcal{F})/\text{Im}(u_n) \to H^i(\mathcal{F}/I^n \mathcal{F}) \to \ker(v_n) \to 0,
\]

we can pass to the inverse limit. Since the transition maps on the left are surjective, we obtain a short exact sequence of \( A \)-modules

\[
0 \to \lim H^i(\mathcal{F})/\text{Im}(u_n) \to \lim H^i(\mathcal{F}/I^n \mathcal{F}) \to \lim \ker(v_n) \to 0.
\]

If we can show that \( \{ \text{Im}(u_n) \} \) cuts out the \( I \)-adic topology on \( H^i(\mathcal{F}) \), then the left side is naturally identified with \( H^i(\mathcal{F}) \otimes_A \hat{A} \) (since \( A \) is noetherian and \( H^i(\mathcal{F}) \) is a finite \( A \)-module). Meanwhile, if we can show that for some large \( d \), the transition maps \( \ker(v_{n+d}) \to \ker(v_n) \)
are 0 for large \( n \), then the right side inverse limit is 0. This will then complete the proof (in particular, we will have proven the ‘topological’ comments as well).

In order to attack the \( I \)-adic topology issue, we begin by readily noting that \( I^nH^i(\mathcal{F}) \subseteq \text{Im}(u_n) \), since \( H^i(\mathcal{F}) \to H^i(\mathcal{F}/I^n\mathcal{F}) \) is \( A \)-linear with kernel \( \text{Im}(u_n) \) and target killed by \( I^n \). We will show that for \( n \geq n_0 \) (for some large \( n_0 \)), \( I \cdot \text{Im}(u_n) = \text{Im}(u_{n+1}) \). This will force

\[
I^nH^i(\mathcal{F}) \subseteq \text{Im}(u_n) \subseteq I^{n-n_0}H^i(\mathcal{F})
\]

for \( n \geq n_0 \). This would establish that the \( \text{Im}(u_n) \)'s cut out the \( I \)-adic topology on \( H^i(\mathcal{F}) \).

Let \( S = \bigoplus_{n \geq 0} I^n \), an \( \mathbb{N} \)-graded ring, finitely generated over \( S_0 = A \) by \( S_1 = I \) (we need to be careful not to confuse \( I^n \subseteq A = S_0 \)). For \( a \in I^m = S_m \), we have the commutative diagram

\[
\begin{array}{ccc}
I^n \mathcal{F} & \xrightarrow{a} & I^{n+m} \mathcal{F} \\
\downarrow & & \downarrow \\
\mathcal{F} & \xrightarrow{a} & \mathcal{F}
\end{array}
\]

and so passing to cohomology gives a commutative diagram

\[
\begin{array}{ccc}
H^i(I^n \mathcal{F}) & \xrightarrow{[a]_m} & H^i(I^{n+m} \mathcal{F}) \\
\downarrow u_n & & \downarrow u_{n+m} \\
H^i(\mathcal{F}) & \xrightarrow{a} & H^i(\mathcal{F})
\end{array}
\]

In other words, the natural action of \( a \in A \) on \( H^i(\mathcal{F}) \) takes \( \text{Im}(u_n) \) into \( \text{Im}(u_{n+m}) \). In this way, \( \bigoplus_{n \geq 0} \text{Im}(u_n) \) is a graded \( S \)-module (where \( S_m = I^m \) just acts via the way \( I^m \subseteq A \) acts on \( H^i(\mathcal{F}) \). If we can prove that this graded \( S \)-module is \textit{finitely generated} over \( S \), then it follows from standard facts about graded modules (discussed long ago) that \( \text{Im}(u_{n+1}) = S_1\text{Im}(u_n) \) for large \( n \geq n_0 \), which is \textit{exactly} what we'd claimed.

In order to prove this finite generatedness, it suffices (by the above commutative diagram) to show that the \( S \)-module \( \bigoplus_{n \geq 0} H^i(I^n \mathcal{F}) \) is finitely generated (since \( \bigoplus \text{Im}(u_n) \) is a quotient as an \( S \)-module). Note that the graded module structure is given by having \( a \in I^m = S_m \) act via \([a]_m \) as in the above diagram. Since \( X \) is a \textit{noetherian} topological space, cohomology commutes with formation of direct limits, so we may study \( H^i(X, \bigoplus I^n \mathcal{F}) \) as an \( S \)-module, where \( S \) acts by functoriality on the cohomology through its action on the sheaf (compatibly with the action of \( A = S_0 \)). Forget about the grading on \( S \) now, but don't forget that \( S \) is a noetherian ring!

Let \( X_S = X \times_{\text{Spec}(A)} \text{Spec}(S) \), so we have an affine projection map \( g : X_S \to X \). Loosely speaking, we have an isomorphism of quasi-coherent \( \mathcal{O}_X \)-algebras \( g_* (\mathcal{O}_{X_S}) \cong \mathcal{O}_X \otimes_A S \) (to be completely precise, we should identify \( S \) with a quasi-coherent sheaf of algebras on \( \text{Spec}(A) \) and pull this back to \( X \), calling the result \( \mathcal{O}_X \otimes_A S \)). Since \( g \) is an affine map, a quasi-coherent \( \mathcal{O}_X \otimes_A S \)-module on \( X \) is ‘the same’ as a quasi-coherent \( \mathcal{O}_{X_S} \)-module on \( X_S \), and is also ‘the same’ as a quasi-coherent \( \mathcal{O}_X \)-module equipped with an action of the ring \( S \) over the action of the ring \( A \) (which acts through \( A \to \Gamma(X, \mathcal{O}_X) \)). Applying this observation to \( \bigoplus I^n \mathcal{F} \) on \( X \), we deduce the existence of a ‘unique’ quasi-coherent \( \mathcal{O}_{X_S} \)-module \( g_* \mathcal{F} \) on \( X_S \) such that \( g_* (\mathcal{F}) \cong \bigoplus I^n \mathcal{F} \) as \( \mathcal{O}_X \otimes_A S \)-modules. Since \( g \) is an affine map, it follows readily (either by considering a degenerate Leray spectral sequence, or else a \v{C}ech cohomology calculation in terms of pullback maps in sheaf cohomology; see Exercise 4.1, Ch III) that
$H^i(X, \oplus I^n\mathcal{F}) \simeq H^i(X_S, \mathcal{G})$ as $S$-modules (due to how the $S$-actions on the two cohomologies can be obtained via functoriality from the action on the sheaves).

We will now see that $\mathcal{G}$ is coherent. Since $X_S \to \text{Spec}(S)$ is proper and $S$ is noetherian, it will then follow that $\oplus H^i(X, I^n\mathcal{F}) \simeq H^i(X_S, \mathcal{G})$ is finitely generated (over $S$)! Consider open affines $\text{Spec}(B)$ in $X$. The open affines $\text{Spec}(B \otimes_A S)$ cover $X_S$ and $\mathcal{G}|_{\text{Spec}(B \otimes_A S)} \simeq \tilde{M}$ for some $B \otimes_A S$-module $M$ (depending on $B$). We just need to show that $M$ is finitely generated over $B \otimes_A S$. If we let $N = \Gamma(\text{Spec}(B), \mathcal{F})$, a finite $B$-module (since $\mathcal{F}$ is coherent), then it is not difficult to check that $M \simeq \oplus_{n \geq 0} I^n N$ as $B \otimes_A S$-modules (where $S = \oplus I^n$ acts in the obvious manner, compatibly with the action of $A = S_0$). Thus, $M$ is a quotient of the $B \otimes_A S$-module $N \otimes_A S$ (via the obvious map), so finiteness of $M$ over $B \otimes_A S$ follows from finiteness of $N$ over $B$.

It remains to find $d$ large so that $\ker(v_{n+d}) \to \ker(v_n)$ is zero for $n$ large. Applying the above considerations for $i+1$ in place of $i$, we see that $\oplus_{n \geq 0} H^{i+1}(I^n\mathcal{F})$ is a finite $S$-module. Since $S$ is noetherian, it follows that the $S$-submodule $\oplus \ker(v_n)$ is also finitely generated (why is this an $S$-submodule?). Using this natural structure of graded $S$-module, there must then exist a large $d$ so that $\ker(v_{n+1}) = S_d \ker(v_n)$ for $n \geq d$. Iterating this $d$ times, and using that $S_d = I^d$, $S_1 = I$, we get $\ker(v_{n+d}) = S_d \ker(v_n)$ for $n \geq d$. We now unwind the definition of $\ker(v_{n+d}) \to \ker(v_n)$ to deduce the vanishing of this map for $n \geq d$.

Consider the inclusion $I^{n+d}\mathcal{F} \hookrightarrow I^n\mathcal{F}$. We have a commutative diagram

$$
\begin{array}{ccc}
\ker(v_{n+d}) & \to & \ker(v_n) \\
\downarrow & & \downarrow \\
H^{i+1}(I^{n+d}\mathcal{F}) & \to & H^{i+1}(I^n\mathcal{F})
\end{array}
$$

where the columns are inclusions. On the other hand, for $a \in I^d = S_d$, the map $a : I^n\mathcal{F} \to I^{n+d}\mathcal{F}$ induces a map $H^{i+1}(I^n\mathcal{F}) \to H^{i+1}(I^{n+d}\mathcal{F})$ which sends $\ker(v_n)$ into $\ker(v_{n+d})$, and which in fact is exactly the map coming from the $S$-module structure on $\oplus \ker(v_n)$. In particular, as $a$ runs through $S_d$, the $A$-linear span of the images of $\ker(v_n) \hookrightarrow H^{i+1}(I^n\mathcal{F}) \xrightarrow{[a]} H^{i+1}(I^{n+d}\mathcal{F})$ for $n \geq d$ is exactly $S_d \ker(v_n) = \ker(v_{n+d})$. The composite of $H^{i+1}(I^n\mathcal{F}) \xrightarrow{[a]} H^{i+1}(I^{n+d}\mathcal{F})$ and $H^{i+1}(I^{n+d}\mathcal{F}) \to H^{i+1}(I^n\mathcal{F})$ (coming from the inclusion of sheaves) is exactly the map $H^{i+1}(I^n\mathcal{F}) \to H^{i+1}(\mathcal{F})$ induced by $a \in I^d \subseteq A = S_0$. Thus, the image of $\ker(v_{n+d}) \to \ker(v_n)$ (which we want to vanish) is exactly the $A$-linear span of the images of the composite maps

$$
\ker(v_n) \hookrightarrow H^{i+1}(I^n\mathcal{F}) \xrightarrow{a} H^{i+1}(I^n\mathcal{F})
$$

for $a \in I^d \subseteq A = S_0$ (and this is the usual $A$-action on the cohomology $H^{i+1}(I^n\mathcal{F})$).

It is therefore necessary and sufficient to show that the ordinary action of $a \in I^d \subseteq A$ on $H^{i+1}(I^n\mathcal{F})$ kills $\ker(v_n)$ for all $n \geq d$. Since $\oplus_{n \geq d} \ker(v_n)$ is generated as an $S$-module by $\ker(v_d)$ (with $S_0 = A$ acting in the usual manner), in order to show that $I^d \subseteq S_0$ kills $\oplus_{n \geq d} \ker(v_n)$, it is enough to show that $I^d \subseteq S_0$ kills $\ker(v_d)$. Now recall that we have an $A$-linear surjection $H^i(\mathcal{F}/I^d\mathcal{F}) \twoheadrightarrow \ker(v_d)$, where the action of $A$ on the right is the same as that of $S_0 = A$. In particular, $a \in I^d \subseteq A = S_0$ kills $\ker(v_d)$, since such $a$ even kills $H^i(\mathcal{F}/I^d\mathcal{F})$. ■