Math 216C. Homework 1

“Ce mémoire, et les nombreux autre qui doivent lui faire suite, sont destinés à former un traité sur les fondements de la Géométrie algébrique. Ils ne présupposent en principe aucune connaissance partielle de cette discipline . . .” EGA I, Introduction.

In Chapter III, do Exercises 6.3+6.7 (do together, giving a counterexample to 6.3(b) if \( \mathcal{F} \) is merely quasi-coherent and \( i = 0 \)), 6.8, 6.9, 6.10, 7.4. Read §§8-§9 of Chapter III.

Extra 1: To what extent does the proof of Serre duality in Hartshorne’s book work over any field \( k \), not assumed to be algebraically closed? What changes in the argument will make it work for any \( k \)?

**Extra 2:** For a scheme \( S \), a relative curve over \( S \) (sometimes called a proper smooth \( S \)-curve) is a finitely presented map \( C \to S \) which is proper and flat with geometric fibers which are connected and regular (by old homework, this latter condition can be checked on any desired algebraically closed extension of each \( k(s) \) for \( s \in S \)).

(i) Can you give an example of a relative curve over \( \mathbb{Z} \) aside from \( \mathbb{P}^1_\mathbb{Z} \)? Can you give an example of a relative curve over \( \mathbb{Q}_p \) which does not ‘extend’ to one over \( \mathbb{Z}_p \)? (even if you can’t give a proof in this latter case, can you at least make a guess?)

(ii) Let \( C \to \text{Spec}(k) \) be a relative curve over a field \( k \). For an extension field \( K/k \), define \( C_K = C \times_k K \).

For any Weil divisor \( D \) on \( C \), define a Weil divisor \( D_K \) on \( C_K \) in such a way that \( D \to D_K \) is additive and such that for composite field extensions \( L/K/k \), \( (D_K)_L = D_L \) as Weil divisors on \( C_L \). Also, using the induced map on function fields \( i : k(C) \to K(C_K) \), show that your definition satisfies \( \text{div}(f)_K = \text{div}(i(f)) \) for \( f \in k(C)^\times \). In particular, you should get a map of class groups \( \text{Cl}(C) \to \text{Cl}(C_K) \) functorial with respect to composite base changes. Check that this map preserves degree and that under the identification of the class group with the Picard group, this “pullback” map becomes exactly the usual pullback map on invertible sheaves.

(iii) Let \( C \to S \) be a relative curve. For an \( S \)-scheme \( T \), define \( \text{Pic}_{\mathcal{L}/S}^0(T) \) to be the subgroup of \( \text{Pic}(C_T) \) consisting of those invertible sheaves on \( C_T = C \times_S T \) which have degree 0 along the fibers over \( T \) (or equivalently — by (iii) — along geometric fibers). Explain how this is a group functor in \( T \), and why this group (for fixed \( T \)) contains the isomorphism classes of all sheaves of the form \( p_T^*(\mathcal{L}) \), where \( p_T : C_T \to T \) is the structure map and \( \mathcal{L} \) is an invertible sheaf on \( T \).

(iv) Let \( C \) be a relative curve over a discrete valuation ring \( A \), with \( K \) the fraction field of \( A \). Prove that the noetherian scheme \( C \) is separated, regular, and connected (hence integral), and use this to prove that the natural “pullback” map \( \text{Pic}(C) \to \text{Pic}(C_K) \) is an isomorphism (note that \( C_K \) is an open subscheme of \( C \), since the generic point of \( \text{Spec}(A) \) is an open point). Using this, show that for a relative curve \( C \to S \) with \( S \) locally noetherian and \( C(S) \) non-empty, and \( \text{Spec}(A) \to S \) any map (with \( A \) a discrete valuation ring and \( K \) its fraction field), the induced map \( \text{Pic}_{\mathcal{L}/S}^0(A) \to \text{Pic}_{\mathcal{L}/S}^0(K) \) is an isomorphism (one needs the existence of a section to \( C \to S \) in order to know that \( \text{Pic}_{\mathcal{L}/S}^0 \) as above is representable; otherwise one needs to alter the functor slightly to get representability and some technical difficulties arise). This turns out to be exactly the valuative criterion proof of the properness of Jacobians.