Math 248A. Sign of discriminant

Let K be a number field, with $n = [K : \mathbf{Q}]$. Let r_1 be the number of embeddings $K \to \mathbf{R}$ and $2r_2$ be the number of non-real embeddings $K \to \mathbf{C}$, so $K \otimes_{\mathbf{Q}} \mathbf{R} \simeq \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$ as $\mathbf{R}$-algebras (by HW1). The aim of this handout is to show that the sign of the nonzero integer $\mathrm{disc}(\mathscr{O}_K/\mathbf{Z}) \in \mathbf{Z}$ is determined by the pair (r_1, r_2). In fact:

Theorem 0.1. *The sign of* $\mathrm{disc}(\mathscr{O}_K/\mathbf{Z})$ *is equal to* $(-1)^{r_2}$.

In the case $[K : \mathbf{Q}] = 2$, this recovers the elementary fact that the discriminant of a quadratic field is positive if and only if it is a real quadratic field.

To prove the theorem, note that unit-square multiples within $\mathbf{Q}^\times$ have no effect on the sign since such squares are positive. Let $q_K : K \to \mathbf{Q}$ be the quadratic form $q_K(x) = \mathrm{Tr}_{K/\mathbf{Q}}(x^2)$, so the associated symmetric $\mathbf{Q}$-bilinear form is $(x, y) \mapsto (1/2)(q_K(x+y) - q_K(x) - q_K(y)) = \mathrm{Tr}_{K/\mathbf{Q}}(xy)$. Thus, a "matrix" that computes this symmetric bilinear form has determinant that is a $(\mathbf{Q}^\times)^2$-multiple of $\mathrm{disc}(\mathscr{O}_K/\mathbf{Z})$ and so our problem is equivalent to computing the sign of such a determinant. This determinant is well-defined (i.e., independent of the choice of $\mathbf{Q}$-basis of K) as an element of $\mathbf{Q}^\times/(\mathbf{Q}^\times)^2$. The natural map $\mathbf{Q}^\times/(\mathbf{Q}^\times)^2 \to \mathbf{R}^\times/(\mathbf{R}^\times)^2 = \langle -1 \rangle$ preserves signs, so it is equivalent to compute the sign of the determinant of a "matrix" that computes the symmetric $\mathbf{R}$-bilinear form associated to the quadratic form $(q_K)_{\mathbf{R}} : K \otimes_{\mathbf{Q}} \mathbf{R} \to \mathbf{R}$ obtained from q_K by the scalar extension $\mathbf{Q} \to \mathbf{R}$.

The formation of the trace map is compatible with scalar extension on the base ring, so $(q_K)_{\mathbf{R}}(u) = \mathrm{Tr}_{K \otimes_{\mathbf{Q}} \mathbf{R}/\mathbf{R}}(u^2)$ for $u \in K \otimes_{\mathbf{Q}} \mathbf{R}$. (If you prefer to work with multilinear expressions rather than with quadratic forms, you can run a similar calculation for the associated symmetric bilinear forms, which is what ultimately matters.) This is a non-degenerate quadratic form on a finite-dimensional $\mathbf{R}$-vector space (with dimension $[K : \mathbf{Q}] = n$), and so by the classification of such quadratic forms it has a signature $(a, n-a)$, which is to say that in a suitable $\mathbf{R}$-linear coordinate system this quadratic form is given by $\sum_{i=1}^{a} y_i^2 - \sum_{j=a+1}^{n} y_j^2$. The matrix of $(q_K)_{\mathbf{R}}$ relative to such a coordinate system is diagonal with a entries equal to 1 and $n - a$ entries equal to -1. Hence, the determinant is $(-1)^{n-a}$ and thus the sign in general is $(-1)^{n-a}$. To summarize, if $(a, n-a)$ denotes the signature of $(q_K)_{\mathbf{R}}$ then the sign of $\mathrm{disc}(\mathscr{O}_K/\mathbf{Z})$ is equal to $(-1)^{n-a}$.

Our problem is now to compute the signature of $(q_K)_{\mathbf{R}}$, and it is given by the following lemma (which obviously implies the Theorem).

Lemma 0.2. *The signature of* $(q_K)_{\mathbf{R}}$ *is* $(r_1 + r_2, r_2)$.

Proof. Recall that $(q_K)_{\mathbf{R}}$ is the quadratic form induced by $\mathrm{Tr}_{K \otimes_{\mathbf{Q}} \mathbf{R}/\mathbf{R}}$. As $\mathbf{R}$-algebras, $K \otimes_{\mathbf{Q}} \mathbf{R} \simeq \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$, and it is easy to check (exercise!) that the distinct factor fields are orthogonal relative to this quadratic form and that the trace map restricted to each factor ring is the $\mathbf{R}$-trace of that factor ring. In particular, if we use an $\mathbf{R}$-basis compatible with this product decomposition of $K \otimes_{\mathbf{Q}} \mathbf{R}$ then the signature of $(q_K)_{\mathbf{R}}$ is the "sum" (as ordered pairs) of the signatures of the induced quadratic forms (which are again $\mathbf{R}$-trace forms) on the factors. In this way we are reduced to showing that $\mathrm{Tr}_{\mathbf{R}/\mathbf{R}}(x^2)$ has signature $(1, 0)$ and $\mathrm{Tr}_{\mathbf{C}/\mathbf{R}}(z^2)$ has signature $(1, -1)$, each of which are easy verifications (e.g., using the $\mathbf{R}$-basis $\{1, i\}$ of $\mathbf{C}$, if $z = x + iy$ with $x, y \in \mathbf{R}$ then $z^2 = (x^2 - y^2) + 2ixy$ has $\mathbf{R}$-trace $2x^2 - 2y^2 = (\sqrt{2}x)^2 - (\sqrt{2}y)^2$). ■