# Math 248A. Homework 8

1. Read §7–§8 in the handout on absolute values.

2. Compute the initial terms of the $p$-adic expansion (up to $a_4p^4$) for $1/7$ and $-3$ in $\mathbf{Q}_p$ for $p = 2, 3, 5$, where the expansions are taken to be in the form $\sum a_n p^n$ with $a_n \in \mathbf{Z}$, $0 \leq a_n < p$. Also compute a square root of 2 in $\mathbf{Q}_p$ for $p \in \{5, 7, 17\}$ up to the term $a_4p^4$, or explain why none exists; be clear on your choice of square root.

3. Let $F$ be a field that is complete with respect to a non-trivial discretely-valued absolute value $|\cdot|$, and let $A$ be the associated valuation ring (so $A$ is a discrete valuation ring). Let $F'/F$ be a finite extension (possibly inseparable!), and let $A'$ be the valuation ring of $F'$ with respect to the unique absolute value $|\cdot|'$ extending $|\cdot|$. Let $k$ and $k'$ be the respective residue fields of $A$ and $A'$.

(*i*) Prove that if $\{a_i'\}$ is a finite set of elements of $A'$ that have $k$-linearly independent images in $k'$, then the $a_i'$'s are $F$-linearly independent in $F'$. Conclude that $[k' : k] \leq [F' : F]$, so $f = [k' : k]$ is *finite*. This is called the *residue field degree* attached to $F'/F$.

(*ii*) Using the norm-formula for $|\cdot|'$ in terms of $|\cdot|$, deduce that $|\cdot|'$ is discretely-valued, with $e \stackrel{\text{def}}{=} [|F'^{\times}| : |F^{\times}|] \leq [F' : F]$, so $A'$ is a discrete valuation ring and $e$ is *finite*; this is called the *ramification degree* attached to $F'/F$. To justify the terminology, show that if $\pi \in A$ is a uniformizer and $\pi' \in A'$ is a uniformizer then $\pi = \pi'^e u'$ with $u' \in A'^{\times}$.

(*iii*) Let $\mathfrak{m}$ and $\mathfrak{m}'$ denote the maximal ideals of $A$ and $A'$. Choose $\{a_i'\}$ in $A'$ lifting a $k$-basis of $k'$. Prove that the $ef$ elements $a_i'\pi'^j$ for $1 \leq i \leq f$ and $0 \leq j \leq e-1$ are $A$-linearly independent, and use $\mathfrak{m}'$-adic completeness of $A'$ and $\mathfrak{m}$-adic completeness of $A$ to prove that the $A$-linear inclusion

$$\bigoplus_{1 \leq i \leq f, 0 \leq j \leq e-1} A a_i' \pi'^j \to A'$$

is an isomorphism (hint: first study $A'/\mathfrak{m}'^r$ for $1 \leq r \leq e$). Hence, $A'$ is a *finite* $A$-module (so $A'$ is the integral closure of $A$ in $F'$) and $[F' : F] = ef$.

4. Let $K$ be a field equipped with a non-trivial non-archimedean absolute value $|\cdot|$ with respect to which the resulting topology is locally compact. Let $A$ be the valuation ring and $\mathfrak{m}$ its maximal ideal (so $\mathfrak{m} \neq 0$ since $|\cdot|$ is non-trivial).

(*i*) Prove that $A$ is open in $K$ and $\mathfrak{m}$ is open in $A$. Also prove that for any $c \in F^{\times}$, $\{x \in F \,|\, |x| \leq |c|\}$ is open and closed, and is homeomorphic to $A$ via $x \mapsto x/c$. Conclude that $F$ is locally compact if and only if $A$ is compact.

(*ii*) Assuming that $A$ is compact, deduce that $\mathfrak{m}$ has *finite* index in $A$, and hence that the residue field $k = A/\mathfrak{m}$ is *finite*. Also prove that $|\cdot|$ must be discretely-valued (so $A$ is a discrete valution ring) and *complete*.

(*iii*) Let $q = \#k$, and let the *normalized absolute value* $\|\cdot\|_K$ be the unique power of $|\cdot|$ with value group $q^{\mathbf{Z}}$ in $\mathbf{R}^{\times}$; that is, $\|\cdot\|_K = q^{-\mathrm{ord}_K}$ where $\mathrm{ord}_K : K^{\times} \twoheadrightarrow \mathbf{Z}$ is the normalized order function (sending uniformizers to 1). Prove that if $\mu$ is a Haar measure on $K$ (which makes sense since $K$ is locally compact) and $a \in K^{\times}$, then $\mu_a(S) = \mu(aS)$ (for Borel sets $S$) is a Haar measure on $K$ and $\mu_a = \|a\|_K \mu$. In other words, the normalized absolute value computes the scaling effect by $K^{\times}$ on Haar measures of $K$.

5. Continuing with the preceding exercise, let $K$ be field that is locally compact with respect to a non-trivial non-archimedean absolute value. Our aim is to classify all such $K$. (The archimedean case was handled in the handout on absolute values.) By Exercise 4, $K$ is complete and its valuation ring $(A, \mathfrak{m})$ is a discrete valuation ring with residue field $k$ that is finite, say with size $q$. Let $p$ denote the characteristic of $k$.

(*i*) Assume $\mathrm{char}(K) > 0$. Prove that $K$ must have characteristic $p$, and prove that the algebraic closure $k_0$ of $\mathbf{F}_p$ in $K$ injects into $k$. Use Hensel's Lemma to prove that in fact $k_0 \to k$ is an isomorphism, so via the inverse (*Teichmüller lifting*) we may canonically view $k$ as a subfield of $K$ (and hence $A$ also has a canonical structure of $k$-algebra).

(*ii*) Continuing with (*i*), upon choosing a uniformizer $\pi$ of $A$ prove that there exists a unique continuous map of $k$-algebras $\phi_\pi : k[\![T]\!] \to A$ sending $T$ to $\pi$ (where $k[\![T]\!]$ is given its $T$-adic topology), and that this map is an *isomorphism* (Hint: use completeness for $K$ to prove surjectivity). Conclude that there is an isometry $K \simeq k(\!(T)\!)$ over $k$ when we use the normalized absolute values on both sides. Explain conversely why $k(\!(T)\!)$ with its $T$-adic topology is locally compact for any finite field, so this gives a classification (up to isomorphism) of local fields with positive characteristic. (It is a general theorem in commutative algebra that if $(A, \mathfrak{m})$ is *any* discrete valuation ring whose fraction field $F$ is $\mathfrak{m}$-adically complete with positive characteristic then there exists an abstract isomorphism of rings $A \simeq k[\![T]\!]$ with $k$ the residue field of $A$, but if $k$ is not algebraic over $\mathbf{F}_p$ then this $k$-algebra structure on $A$ is *not* canonical.)

(*iii*) Now assume $K$ has characteristic 0, so $p \in \mathfrak{m}$ is nonzero. Let $e = \mathrm{ord}_K(p) > 0$ and let $f = [k : \mathbf{F}_p]$. Let $\pi$ be a uniformizer of $A$. Explain why $K$ is canonically an extension of $\mathbf{Q}_p$, with $\|\cdot\|_K$ restricting to $|\cdot|_p^{ef}$, and prove that if $\{a_i\}$ in $A$ is a set with $\mathbf{F}_p$-linearly independent image in $k$ then the $a_i$'s are $\mathbf{Q}_p$-linearly independent in $K$.

(*iv*) Continuing with (*iii*), let $\{a_i\}$ in $A$ lift an $\mathbf{F}_p$-basis of $k$. Prove that $\pi^e = pu$ with $u \in A^\times$, and deduce (via $\pi$-adic completeness of $A$ and $p$-adic completeness of $\mathbf{Z}_p$) that the natural $\mathbf{Z}_p$-linear map

$$\bigoplus_{1 \le i \le f, 0 \le j \le e-1} a_i \pi^j \mathbf{Z}_p \to A$$

is an isomorphism. Conclude that $[K : \mathbf{Q}_p] = ef$ is *finite*. Conversely, prove that any finite extension of $\mathbf{Q}_p$ (endowed with its canonical topology using the absolute value extending that on $\mathbf{Q}_p$) is a locally compact field. Hence, the non-archimedean local fields with characteristic 0 are precisely the finite extensions of the $p$-adic fields $\mathbf{Q}_p$.