Math 248B. Complex structure and Shimura data

1. Motivation

As we have seen, the natural action of $\mathrm{GL}_2(\mathbf{R})$ on $\mathbf{C}-\mathbf{R}$ identifies $\mathbf{C}-\mathbf{R}$ real-analytically with $\mathrm{GL}_2(\mathbf{R})/Z(\mathbf{R})K_\infty^0$ where K_∞ is a maximal compact subgroup of $\mathrm{GL}_2(\mathbf{R})$ whose identity component K_∞^0 is a maximal compact subgroup of $\mathrm{SL}_2(\mathbf{R})$. This identification depends on a choice of base point $\tau_0 \in \mathbf{C}-\mathbf{R}$, the choice of which amounts to selecting $i=\sqrt{-1}\in\mathbf{C}$ and a maximal compact subgroup K_∞ (the link being that τ_0 is the unique point in the i-component of $\mathbf{C}-\mathbf{R}$ having K_∞^0 as its stabilizer in $\mathrm{SL}_2(\mathbf{R})$). The choice of τ_0 doesn't matter in the following sense. Let $\phi_{\tau_0}:\mathrm{GL}_2(\mathbf{R})/Z(\mathbf{R})K_\infty^0 \simeq \mathbf{C}-\mathbf{R}$ be the real-analytic isomorphism defined by the orbit map $g\mapsto [g](\tau_0)$ at τ_0. Any other choice τ_1 has the form $\tau_1=[g_0](\tau_0)$ for some $g_0\in\mathrm{GL}_2(\mathbf{R})$ and corresponding maximal compact subgroup $g_0K_\infty^0g_0^{-1}$ in its stabilizer. The orbit maps ϕ_{τ_0} and ϕ_{τ_1} fit into the commutative diagram

$$\begin{array}{ccc} \mathrm{GL}_2(\mathbf{R})/Z(\mathbf{R})K_\infty^0 & \xrightarrow{\phi_{\tau_0}} & \mathbf{C}-\mathbf{R} \\ {\scriptstyle g\mapsto g_0gg_0^{-1}}\downarrow & & \downarrow{\scriptstyle [g_0]} \\ \mathrm{GL}_2(\mathbf{R})/Z(\mathbf{R})(g_0K_\infty^0g_0^{-1}) & \xrightarrow[\phi_{\tau_1}]{} & \mathbf{C}-\mathbf{R} \end{array}$$

in which the right side is a holomorphic automorphism and the other maps are real-analytic isomorphisms. Hence, the complex-analytic structure imposed on $\mathrm{GL}_2(\mathbf{R})/Z(\mathbf{R})K_\infty^0$ is "independent of the choice of τ_0" (equivalently, the choice of $i=\sqrt{-1}\in\mathbf{C}$ and K_∞^0).

This procedure endows the quotient $\mathrm{GL}_2(\mathbf{R})/Z(\mathbf{R})K_\infty^0$ with a complex-analytic structure that is also invariant with respect to the left $\mathrm{GL}_2(\mathbf{R})$-action (i.e., the left-translation action by each $g\in\mathrm{GL}_2(\mathbf{R})$ is through a holomorphic automorphism, in fact going over to $[g]$ on $\mathbf{C}-\mathbf{R}$). It is natural to wonder what features of $\mathrm{GL}_2(\mathbf{R})$ underlie this phenomenon, since it is certainly not true in general for interesting connected linear algebraic $\mathbf{R}$-groups G (such as GL_n, SL_n, etc.) with center Z and maximal compact subgroup K in $G(\mathbf{R})$ that $G(\mathbf{R})/Z(\mathbf{R})K^0$ admits a $G(\mathbf{R})$-invariant holomorphic structure, or any holomorphic structure at all. For example, this coset space can have odd dimension, such as for $G=\mathrm{GL}_n$ (with maximal compact subgroup $\mathrm{O}_n(\mathbf{R})$ of dimension $n(n-1)/2$, so $G(\mathbf{R})/Z(\mathbf{R})K^0$ has dimension $n^2-1-n(n-1)/2$ when $n\equiv 0,3 \bmod 4$.

There is a general framework (due to Deligne) which explains the source of such holomorphic structures (by a construction that is essentially a vast generalization of $\mathbf{C}-\mathbf{R}=\mathbf{P}^1(\mathbf{C})-\mathbf{P}^1(\mathbf{R})$ for $G=\mathrm{GL}_2$), and in this handout we explain how it goes. For some of the later arguments we will need to appeal to results from the theory of reductive groups (the general construction of complex structures requires it, as do the deeper aspects of the theory), though for the GL_2-case everything will be seen by direct arguments.

Notation. For a finite-dimensional vector space V over a field k, we write $\mathrm{GL}(V)$ to denote the k-group whose R-points are $\mathrm{Aut}_R(V\otimes_k R)$ for any k-algebra R (functorially in R); for example, $\mathrm{GL}(k^n)=\mathrm{GL}_n$. When we wish to refer to its group of k-points, we write $\mathrm{Aut}(V)$. For a field extension k'/k, we sometimes write $V_{k'}$ to denote the k'-vector space $k'\otimes_k V$.

For a complex manifold M with underlying C^∞-manifold $M_{\mathbf{R}}$, the tangent space $\mathrm{T}_m(M_{\mathbf{R}})$ at a point $m\in M_{\mathbf{R}}$ is naturally identified with the underlying $\mathbf{R}$-vector space of the tangent space $\mathrm{T}_m(M)$. This is traditionally defined via coordinate formulas, but can also be explained conceptually. See the Appendix.

2. Almost complex structures

Rather generally, if X and Y are complex manifolds then a C^∞ map $f:X_{\mathbf{R}}\to Y_{\mathbf{R}}$ between the underlying smooth manifolds is holomorphic if and only if the tangent maps $\mathrm{d}f(x):\mathrm{T}_x(X_{\mathbf{R}})\to\mathrm{T}_{f(x)}(Y_{\mathbf{R}})$ are $\mathbf{C}$-linear for all $x\in X_{\mathbf{R}}$ (relative to the $\mathbf{C}$-linear structure arising from the holomorphic structure), as this encodes the Cauchy–Riemann equations in local holomorphic coordinates (see (A.1) in the Appendix). Applying this to the identity map of a C^∞ manifold, it follows that for a complex manifold X, the holomorphic structure on the underlying C^∞ manifold $X_{\mathbf{R}}$ is *uniquely determined* by the $\mathbf{C}$-linear structure on each $\mathrm{T}_x(X_{\mathbf{R}})$, or equivalently the linear automorphisms $J_x : \mathrm{T}_x(X_{\mathbf{R}}) \to \mathrm{T}_x(X_{\mathbf{R}})$ arising from multiplication by a fixed choice of $i = \sqrt{-1} \in \mathbf{C}$ on each tangent space. The automorphisms J_x constitutes a C^∞ automorphism J of the tangent bundle $\mathrm{T}(X_{\mathbf{R}}) \to X$ satisfying $J^2 = -1$. In general for a C^∞ manifold M an *almost complex structure* on M is a C^∞ automorphism J of $\mathrm{T}M \to M$ such that $J^2 = -1$, and it is called *integrable* (or a *complex structure*) if it arises from a complex-analytic structure on M (which we have seen is unique if it exists).

Example 2.1. Fix a choice of $i = \sqrt{-1} \in \mathbf{C}$ and identify $\mathbf{C}$ with $\mathbf{R}^2$ via the ordered basis $\{i, 1\}$ (for consistency with classical linear fractional transformation formulas). Consider $M = \mathrm{GL}_2(\mathbf{R})/Z(\mathbf{R})\mathrm{SO}_2(\mathbf{R})$. Define

$$r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix};$$

this is the matrix for multiplication by $e^{i\theta}$ relative to the basis $\{i, 1\}$ of $\mathbf{C}$ (not $\{1, i\}$; note the sign in the lower-left, not upper-right!). The group $K_i := \mathrm{SO}_2(\mathbf{R})$ of such rotations in $\mathrm{GL}_2(\mathbf{R})$ is the stabilizer of i, and we use K_i as our preferred maximal compact subgroup of $\mathrm{SL}_2(\mathbf{R})$ in order to simplify our calculations below.

Identify M with $\mathbf{C} - \mathbf{R}$ via $g \mapsto [g](i)$, so $g = \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$ is carried to $yi + x = x + iy$ for $y \in \mathbf{R}^\times$. Thus, the multiplication map

$$\left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right\} \times Z(\mathbf{R})\mathrm{SO}_2(\mathbf{R}) \to \mathrm{GL}_2(\mathbf{R})$$

is a C^∞-isomorphism, and in this way we may and do view $\{y, x\}$ as global C^∞ coordinates on M. In particular, $\{\partial_y, \partial_x\}$ is a global ordered frame for $\mathrm{T}(M)$.

Let $m_0 \in M$ be the coset class of the identity in $\mathrm{GL}_2(\mathbf{R})$, which goes over to our chosen $i = \sqrt{-1} \in \mathbf{C} - \mathbf{R}$. The $\mathrm{GL}_2(\mathbf{R})$-invariant complex structure on M via the identification with $\mathbf{C} - \mathbf{R}$ yields a $\mathrm{GL}_2(\mathbf{R})$-invariant almost complex structure J on M whose effect on $\mathrm{T}_{m_0}(M)$ is $J_{m_0}(\partial_y|_{m_0}) = -\partial_x|_{m_0}$ and $J_{m_0}(\partial_x|_{m_0}) = \partial_y|_{m_0}$. In view of the transitivity of the left $\mathrm{GL}_2(\mathbf{R})$-action on M, to specify J is exactly to specify J_{m_0}, so to give a Lie-theoretic characterization of the complex structure on M it is equivalent to give a Lie-theoretic construction of J_{m_0}. We now do this.

Conjugation $g \mapsto r_\theta g r_\theta^{-1}$ on $\mathrm{GL}_2(\mathbf{R})$ preserves $Z(\mathbf{R})K_i$, so it induces an automorphism of M fixing m_0 and hence an automorphism of $\mathrm{T}_{m_0}(M)$. Let's compute the effect relative to the ordered basis $\{\partial_y|_{m_0}, \partial_x|_{m_0}\}$. The right multiplication by r_θ^{-1} is irrelevant when working modulo $Z(\mathbf{R})K_i$, so we can work with the left translation $g \mapsto r_\theta g$ for $g = \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$ (noting that r_θ acting on $M = \mathrm{GL}_2(\mathbf{R})/Z(\mathbf{R})K_i \simeq \mathbf{C} - \mathbf{R}$ does indeed preserve $m_0 = i$). We'll use the presentation of the coset space M as $\mathbf{C} - \mathbf{R}$ via linear fractional action on i in order to most easily read off the coordinates of the coset class of $r_\theta g$ in M. This presentation describes the map as

$$x + iy = [g](i) \mapsto [r_\theta g](i) = \frac{(x\cos\theta + \sin\theta) + iy\cos\theta}{(-x\sin\theta + \cos\theta) - iy\sin\theta}.$$

To compute the image of $\partial_x|_{m_0}$ we restrict to the 1-parameter path $t \mapsto t + i \cdot 0$ through $(0, 1)$ and compute the image of $\partial_t|_0$. This restriction is

$$\sigma_x : t \mapsto \frac{(t\cos\theta + \sin\theta) + i\cos\theta}{-t\sin\theta + e^{-i\theta}}.$$

Differentiation with respect to t yields $1/(-t\sin\theta + e^{-i\theta})^2$, so evaluating at $t = 0$ gives $e^{2i\theta} = i\sin(2\theta) + \cos(2\theta)$. Similarly, restricting to the 1-parameter path $t \mapsto 0 + it$ through $(0, 1)$ and computing the image of $\partial_t|_1$ goes as follows: the restriction is

$$\sigma_y : t \mapsto \frac{\sin\theta + t\cos\theta i}{\cos\theta - t\sin\theta i},$$

and differentiation yields $i/(\cos\theta - t\sin\theta i)^2$, so evaluating at $t = 1$ gives $ie^{2i\theta} = i\cos(2\theta) - \sin(2\theta)$.

Summarizing, the effect of r_θ-conjugation on $\mathrm{T}_{m_0}(M)$ has the matrix

$$\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{pmatrix} = r_{2\theta}$$

with respect to the ordered basis $\{\partial_y|_{m_0}, \partial_x|_{m_0}\}$ (not $\{\partial_x|_{m_0}, \partial_y|_{m_0}\}$!). Hence, if we take $\theta = 2\pi/8$ then we get $r_{\pi/4}$, which is the usual "rotation by i" relative to our coordinatization of $\mathbf{C} - \mathbf{R}$ (via the ordered $\mathbf{R}$-basis $\{i, 1\}$ of $\mathbf{C}$). In other words, the classical $\mathrm{GL}_2(\mathbf{R})$-invariant complex structure on M arises from the unique $\mathrm{GL}_2(\mathbf{R})$-invariant almost complex structure defined by taking J_{m_0} to be the effect on $\mathrm{GL}_2(\mathbf{R})$ by $g \mapsto h_i(e^{2\pi i/8}) g h_i(e^{2\pi i/8})^{-1}$, where $h_i : \mathbf{C} \to \mathrm{Mat}_2(\mathbf{R})$ is the $\mathbf{R}$-algebra embedding defined by the ordered $\mathbf{R}$-basis $\{i, 1\}$ of $\mathbf{C}$.

Note that since $h_i(\mathbf{C}^\times) = Z(\mathbf{R})K_i$ and this commutative, it is clear by direct inspection that the effect on $\mathrm{Lie}(\mathrm{GL}_2(\mathbf{R}))$ of conjugation on $\mathrm{GL}_2(\mathbf{R})$ by any $h_i(z)$ induces an automorphism of $\mathrm{T}_{m_0}(M)$ that uniquely propagates to a $\mathrm{GL}_2(\mathbf{R})$-invariant automorphism of $\mathrm{T}(M)$. In particular, we can see directly that using $z = e^{2\pi i/8}$ yields an almost complex structure J on M. What is *not* obvious without appealing to our earlier independent construction of a complex structure on M and checking its agreement with J (via $\mathrm{GL}_2(\mathbf{R})$-equivariance and a calculation at m_0) is that the almost complex structure J really is integrable. We have not yet truly understood what is going on (in a manner which can be applied to more general Lie groups).

For a given almost complex structure on a smooth manifold, the deep *Newlander-Nirenberg theorem* gives a computable necessary and sufficient condition for its integrability (the criterion is the vanishing of a certain tensor field associated to an almost complex structure). The hard part is the sufficiency of the criterion (since constructing a global holomorphic structure is generally a difficult task). As an example, on S^6 there is a non-integrable almost complex structure arising from the identification of S^6 with the set of pure imaginary octonions (an $\mathbf{R}^7$) with norm 1, and it is a famous open problem to determine whether S^6 admits a complex structure. This is particularly interesting since it is known that S^2 and S^6 are the only spheres (with their unique C^∞-structure!) which admit an almost complex structure; there are topological obstructions for all other even-dimensional spheres.

In Example 2.1, we used an independent construction of the complex structure on $\mathrm{GL}_2(\mathbf{R})/Z(\mathbf{R})K_i$ via $\mathbf{C} - \mathbf{R}$ to prove that the $\mathrm{GL}_2(\mathbf{R})$-invariant almost complex structure determined by the effect of $h_i(e^{2\pi i/8})$-conjugation on $\mathrm{Lie}(\mathrm{GL}_2(\mathbf{R}))$ actually integrates to a complex structure on $\mathrm{GL}_2(\mathbf{R})/Z(\mathbf{R})K_i$. To understand why this works in a more conceptual way, we need to take a different point of view. We do *not* appeal to the Newlander-Nirenberg theorem (as that is rather inconvenient to use for our purposes).

3. Centralizers, normalizers, and S^1-actions

In this section let $G = \mathrm{GL}_2$ and $\mathfrak{g} = \mathfrak{gl}_2(\mathbf{R}) = \mathrm{Lie}(G(\mathbf{R}))$. Any two $\mathbf{R}$-algebra embeddings $h, h' : \mathbf{C} \rightrightarrows \mathrm{Mat}_2(\mathbf{R})$ are $\mathrm{GL}_2(\mathbf{R})$-conjugate (since such an embedding equips $\mathbf{R}^2$ with a structure of 1-dimensional $\mathbf{C}$-vector space, and any two such structures are $\mathbf{C}$-linearly isomorphic; alternatively, as overkill apply the Skolem–Noether theorem). Note that $h(\mathbf{C}^\times)$ is its own centralizer in $G(\mathbf{R})$, since an automorphism of a 1-dimensional $\mathbf{C}$-vector space must be multiplication by some element of $\mathbf{C}^\times$. The subgroups $h(\mathbf{C}^\times)$ in $G(\mathbf{R})$ are called the *non-split Cartan* subgroups.

Lemma 3.1. *The normalizer $N_{G(\mathbf{R})}(h(\mathbf{C}^\times))$ contains $h(\mathbf{C}^\times)$ with index 2. Explicitly, $N_{G(\mathbf{R})}(h(\mathbf{C}^\times)) = \mathbf{R}^\times K_\infty$ for the unique maximal compact subgroup K_∞ whose identity component K_∞^0 is the circle in $h(\mathbf{C}^\times)$, and the conjugation action on $h(\mathbf{C})$ by the nontrivial element in $N_{G(\mathbf{R})}(h(\mathbf{C}^\times))/h(\mathbf{C}^\times)$ is $h(z) \mapsto h(\overline{z})$.*

Proof. This lemma can be proved by explicit calculation after conjugating h into a special case (such as carrying i to $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ via viewing $\{i, 1\}$ as an $\mathbf{R}$-basis of $\mathbf{C}$). We leave that method to the interested reader, and offer a conceptual proof instead.

Since $z \mapsto h(\overline{z})$ is another $\mathbf{R}$-algebra embedding of $\mathbf{C}$ into $\mathrm{Mat}_2(\mathbf{R})$, there exists $g_0 \in G(\mathbf{R})$ such that $h(\overline{z}) = g_0 h(z) g_0^{-1}$. Thus, g_0-conjugation normalizes $h(\mathbf{C}^\times)$ but does not centralize it, so $N_{G(\mathbf{R})}(h(\mathbf{C}^\times))$ strictly contains $h(\mathbf{C}^\times)$.

Consider an arbitrary element $g \in N_{G(\mathbf{R})}(h(\mathbf{C}^\times))$ not in $h(\mathbf{C}^\times) = Z_{G(\mathbf{R})}(h(\mathbf{C}^\times))$, so g-conjugation acts nontrivially on $h(\mathbf{C}^\times)$. Since g-conjugation on $\mathrm{Mat}_2(\mathbf{R})$ is an $\mathbf{R}$-algebra automorphism, so by preservation of $h(\mathbf{C}^\times)$ it follows that g-conjugation restricts to an $\mathbf{R}$-algebra automorphism of $h(\mathbf{C}) = h(\mathbf{C}^\times) \cup \{0\}$. This is a nontrivial automorphism (since we chose g not to lie in $Z_{G(\mathbf{R})}(h(\mathbf{C}^\times)) = h(\mathbf{C}^\times)$), and so it must be $h(z) \mapsto h(\overline{z})$ because complex conjugation is the unique nontrivial $\mathbf{R}$-algebra automorphism of $\mathbf{C}$. Hence, any two choices $g, g' \in N_{G(\mathbf{R})}(h(\mathbf{C}^\times))$ not in $h(\mathbf{C}^\times)$ have the same conjugation effect on $h(\mathbf{C}^\times)$, so $g'g^{-1} \in Z_{G(\mathbf{R})}(h(\mathbf{C}^\times)) = h(\mathbf{C}^\times)$. This implies that $h(\mathbf{C}^\times)$ has index 2 in its $G(\mathbf{R})$-normalizer.

To prove the explicit description of the normalizer in terms of compact subgroups of $G(\mathbf{R})$, consider the description $h(\mathbf{C}^\times) = \mathbf{R}^\times K_\infty^0$ in terms of the identity component K_∞^0 (a circle) of a maximal compact subgroup K_∞ in $G(\mathbf{R})$. This is certainly normalized by K_∞, so for index reasons $N_{G(\mathbf{R})}(h(\mathbf{C}^\times)) = \mathbf{R}^\times K_\infty$. But any subgroup of $G(\mathbf{R})$ containing K_∞^0 with index 2 must normalize it, and for dimension reasons any maximal compact subgroup K of $G(\mathbf{R})$ containing K_∞^0 must have K_∞^0 as its identity component, so K normalizes K_∞^0 and therefore is contained in the normalizer $\mathbf{R}^\times K_\infty$ of $h(\mathbf{C}^\times) = \mathbf{R}^\times K_\infty^0$ (as $\mathbf{R}^\times$ is central in $G(\mathbf{R})$). By inspection, every compact subgroup of $\mathbf{R}^\times K_\infty$ is contained in K_∞ (as the central -1 lies in K_∞, and the quotient $\mathbf{R}^\times K_\infty / K_\infty = \mathbf{R}^\times / \langle -1 \rangle = \mathbf{R}^\times_{>0}$ contains no nontrivial compact subgroups), so $K \subseteq K_\infty$ and hence $K = K_\infty$ by maximality. ■

When $\mathbf{R}^\times$ is viewed as a Lie group (rather than as the $\mathbf{R}$-points of the algebraic group GL_1), it acquires automorphisms which do not arise from algebraic geometry. Indeed, $\mathbf{R}^\times = \langle -1 \rangle \times \mathbf{R}^\times_{>0}$ and we can use the map $t \mapsto t^y$ on $\mathbf{R}^\times_{>0}$ for any $y \in \mathbf{R}$. To avoid such exotic maps, we are led to:

Definition 3.2. For a linear algebraic group H over $\mathbf{R}$ (i.e., a smooth affine $\mathbf{R}$-group), a homomorphism of Lie groups $h : \mathbf{C}^\times \to H(\mathbf{R})$ is *algebraic* if its restriction to $\mathbf{R}^\times$ arises from a (necessarily unique) map of $\mathbf{R}$-groups $\mathrm{GL}_1 \to H$.

This definition certainly captures the maps arising from $\mathbf{R}$-algebra embeddings of $\mathbf{C}$ into $\mathrm{Mat}_2(\mathbf{R})$ (taking $H = G = \mathrm{GL}_2$), but it may look slightly ad hoc. The formulation of the definition is given to avoid more background in algebraic groups. A more insightful definition is given in the following remark.

Remark 3.3. For the reader who is familiar with the theory of algebraic groups, Weil restriction of scalars, and compact Lie groups (including its underlying algebraicity), here is an equivalence with a more natural definition. Observe that $\mathbf{C}^\times$ is generated by $\mathbf{R}^\times$ and S^1 with intersection $\langle -1 \rangle$, and the $\mathbf{R}$-torus

$$\mathbf{S} := \mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathrm{GL}_1)$$

(sometimes called the *Deligne torus*) is generated by its $\mathbf{R}$-subgroups GL_1 and $\ker(\mathrm{N}_{\mathbf{C}/\mathbf{R}} : \mathbf{S} \to \mathrm{GL}_1)$ with intersection μ_2. Also, since every compact subgroup of $\mathrm{GL}_n(\mathbf{R})$ is contained in the subgroup $\mathrm{O}_n(\mathbf{R})$ up to conjugacy, any map from a connected compact Lie group (such as S^1) to the $\mathbf{R}$-points of a linear algebraic $\mathbf{R}$-group is necessarily algebraic (relative to the equivalence between connected compact Lie groups and anisotropic connected reductive $\mathbf{R}$-groups). Hence, a Lie group map $h : \mathbf{C}^\times = \mathbf{S}(\mathbf{R}) \to H(\mathbf{R})$ is algebraic in the sense of Definition 3.2 if and only if h arises from a (necessarily unique) map of $\mathbf{R}$-groups $\mathbf{S} \to H$.

We have seen above that within the set $\mathrm{Hom}_{\mathrm{alg}}(\mathbf{C}^\times, G(\mathbf{R}))$ of algebraic Lie group homomorphisms for $G = \mathrm{GL}_2$ there is a distinguished $G(\mathbf{R})$-conjugacy class, namely those arising from $\mathbf{R}$-algebra embeddings of $\mathbf{C}$ into $\mathrm{Mat}_2(\mathbf{R})$. There are further Lie-theoretic properties that these homomorphisms satisfy (expressed in terms of $G(\mathbf{R})$ without reference to $\mathrm{Mat}_2(\mathbf{R})$), as we now explain.

Consider the composite map

$$\mathbf{C}^\times \xrightarrow{h} G(\mathbf{R}) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g})$$

where $\mathrm{Ad}(g) = \mathrm{Lie}(c_g)$ denotes the "adjoint action" of $G(\mathbf{R}) = \mathrm{GL}_2(\mathbf{R})$ on $\mathfrak{g} = \mathrm{Lie}(G(\mathbf{R})) = \mathfrak{gl}_2(\mathbf{R}) = \mathrm{Mat}_2(\mathbf{R})$ (defined as the automorphism of $\mathfrak{g}$ induced by the conjugation automorphism $c_g : x \mapsto gxg^{-1}$ of G). Concretely, for $\mathrm{GL}_n(\mathbf{R})$ the adjoint action $\mathrm{Ad} : \mathrm{GL}_n(\mathbf{R}) \to \mathrm{Aut}(\mathfrak{gl}_n(\mathbf{R}))$ carries any $T \in \mathrm{GL}_n(\mathbf{R})$ to the conjugation automorphism $M \mapsto TMT^{-1}$ of $\mathfrak{gl}_n(\mathbf{R}) = \mathrm{Mat}_n(\mathbf{R})$, so in our setting with GL_2 we are considering the conjugation action on $\mathrm{Mat}_2(\mathbf{R})$ by $h(z)$ for $z \in \mathbf{C}^\times$.

To understand how $\mathbf{C}^\times$ acts on $\mathfrak{g} = \mathrm{Mat}_2(\mathbf{R})$ through h, it is convenient to separate this into two problems: the action of $\mathbf{R}^\times$ and the action of S^1. For the $\mathbf{R}^\times$-action things are straightforward: the h's that we care about carry $\mathbf{R}^\times$ into the center $\mathbf{R}^\times$ of $G(\mathbf{R})$ via the identity map. In particular, $h(\mathbf{R}^\times)$ has trivial conjugation action on $G(\mathbf{R})$ and thus has trivial adjoint action on $\mathfrak{g}$. This implies that the element $-1 \in \mathbf{R}^\times \cap S^1$ acts trivially on $\mathfrak{g}$. To give a conceptual description of the S^1-action, we will use that S^1-actions on $\mathbf{C}$-vector spaces can be canonically diagonalized:

Proposition 3.4. *Let V be a finite-dimensional $\mathbf{C}$-vector space equipped with a continuous linear action by S^1. For each $r \in \mathbf{Z}$, define V_r to be the subspace of $v \in V$ such that $z.v = z^r v$ for all $z \in S^1$. The natural map $\oplus_r V_r \to V$ is an isomorphism. In particular, $V_r = 0$ for all but finitely many r.*

The subspace V_r is called the *r-weight space* for the S^1-action; e.g. V_0 is the set of fixed-points for the action.

Proof. The torsion elements of S^1 act with finite order and pairwise commute, so they can be simultaneously diagonalized. In other words, upon choosing a suitable basis, we can assume that the dense subgroup μ_∞ of roots of unity has diagonal image in $\mathrm{GL}_n(\mathbf{C})$. By continuity, S^1 also has diagonal image and the resulting diagonal characters are continuous homomorphisms $S^1 \to \mathbf{C}^\times = \mathbf{R}^\times_{>0} \times S^1$. These arise from continuous homomorphisms $S^1 \to S^1$, and it is classical that the only such homomorphisms are the power maps $z \mapsto z^r$ for $r \in \mathbf{Z}$. The desired assertions are now easily verified. ∎

We conclude that the S^1-action on $V = \mathfrak{g}_\mathbf{C} = \mathfrak{gl}_2(\mathbf{C})$ has a weight-space decomposition $\oplus_r V_r$. The integers r such that $V_r \neq 0$ (called the *weights* of the action) must satisfy $(-1)^r v = 0$ for all $v \in V_r$ (since $-1 \in \mathbf{R}^\times$ acts trivially), so all such r are even. We can do much better:

Lemma 3.5. *The weight spaces for the S^1-action on the 4-dimensional $V = \mathfrak{g}_\mathbf{C}$ are V_0 of dimension 2 and $V_{\pm 2}$ each of dimension 1.*

Proof. The closed subgroup $h(\mathbf{C}^\times) \subseteq G(\mathbf{R})$ commutes with $h(S^1)$, so its 2-dimensional Lie algebra in $\mathfrak{g}$ is centralized by the adjoint action of S^1. Hence, the 2-dimensional $\mathbf{C}$-subspace $\mathrm{Lie}(h(\mathbf{C}^\times))_\mathbf{C} \subseteq \mathfrak{g}_\mathbf{C} = V$ is contained in the weight-0 space V_0.

In general, for W of dimension $d > 0$ the left-translation action $M \mapsto TM$ on $\mathrm{End}(W)$ by a point T of $\mathrm{GL}(W)$ is a linear endomorphism of $\mathrm{End}(W)$ whose determinant is $(\det T)^d$, and similarly for right-translation. (To verify these assertions, note that they are universal polynomial identities. Thus, we can reduce to the case of diagonalizable T, for which we can compute relative to a W-basis of T-eigenvectors.) Hence, the adjoint action by $\mathrm{GL}(W)$ on the vector space $\mathrm{End}(W)$ has *trivial* determinant, so the sum of all weights (with multiplicity) for the S^1-action on $\mathfrak{g}_\mathbf{C}$ is 0. But nonzero weights *do* occur, since $G(\mathbf{C})$ is connected with $h(S^1)$ noncentral (so the adjoint action by $h(S^1)$ on $\mathrm{Lie}(G(\mathbf{C})) = \mathfrak{g}_\mathbf{C} = V$ must be nontrivial). Hence, the weight space decomposition for V relative to the S^1-action must have the form $V_0 \oplus V_r \oplus V_{-r}$ for a 2-dimensional V_0 and lines $V_{\pm r}$ for some $r > 0$.

To prove that $r = 2$, we first note that for $z \in S^1$ the element $h(z) \in G(\mathbf{R})$ is diagonalizable over $\mathbf{C}$ with eigenvalues $\{z, 1/z\}$ (since h comes from an $\mathbf{R}$-algebra embedding $\mathbf{C} \hookrightarrow \mathrm{Mat}_2(\mathbf{R})$ and $1/z$ is the $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-conjugate of $z \in S^1$). To compute the eigenvalues for the conjugation action by $h(z)$ on $\mathrm{Mat}_2(\mathbf{C}) = \mathrm{End}(\mathbf{C}^2)$ it is harmless to replace $h(z)$ with any $\mathrm{GL}_2(\mathbf{C})$-conjugate, or correspondingly we can make any change of basis on $\mathbf{C}^2$ that we wish. Using a $\mathbf{C}$-basis of eigenvectors for $h(z)$ then makes the $h(z)$-action into conjugation by $\left(\begin{smallmatrix} z & 0 \\ 0 & 1/z \end{smallmatrix}\right)$ on $\mathrm{Mat}_2(\mathbf{C})$. Since

$$\begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix}^{-1} = \begin{pmatrix} a & z^2 b \\ z^{-2} c & d \end{pmatrix},$$

we see by inspection that the adjoint action of $h(z)$ has as eigenavalues $1, z^2, z^{-2}$: when $z^2 \neq 1$, the z^2-eigenspace is $\{\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)\}$ and the z^{-2}-eigenspace is $\{\left(\begin{smallmatrix} 0 & 0 \\ * & 0 \end{smallmatrix}\right)\}$. ∎

Summarizing, the homomorphisms $h : \mathbf{C}^\times \to G(\mathbf{R})$ arising from $\mathbf{R}$-algebra embeddings $\mathbf{C} \to \mathrm{Mat}_2(\mathbf{R})$ are algebraic and carry $\mathbf{R}^\times$ into the center via the identity map, with the resulting adjoint action of S^1 on $\mathfrak{g}_\mathbf{C}$ having as its weights exactly 0 (with multiplicity 2) and ± 2 (each with multiplicity 1). The reader can check that these properties *characterize* such h among all Lie group maps $\mathbf{C}^\times \to G(\mathbf{R})$. To make sense of these conditions in a broader context, with GL_2 replaced by a more general linear algebraic $\mathbf{R}$-group (thereby losing contact with anything like the $\mathbf{R}$-algebra $\mathrm{Mat}_2(\mathbf{R})$ whose unit group is $\mathrm{GL}_2(\mathbf{R})$), we need to introduce a new concept: Hodge structures. This is the topic of the next section.

4. Hodge structures

Our goal is to introduce Deligne's concept of a *Shimura datum*, as this will put the preceding case of GL_2 into a broader framework (to better understand the source of the complex structure on $\mathrm{GL}_2(\mathbf{R})/\mathbf{R}^\times K_\infty^0$). Since we will not be focusing exclusively on the special case of GL_2 (though we will often return to it in examples), in this section we use the notation "G" with more general meaning.

Let G be a connected linear algebraic group over $\mathbf{R}$, and consider maps of Lie groups $h : \mathbf{C}^\times \to G(\mathbf{R})$ that are algebraic in the sense of Definition 3.2. There is a natural conjugation action $(g, h) \mapsto g.h$ of $G(\mathbf{R})$ on the set $\mathrm{Hom}_{\mathrm{alg}}(\mathbf{C}^\times, G(\mathbf{R}))$ of such "algebraic" homomorphisms (where $(g.h)(z) = gh(z)g^{-1}$), and we fix a $G(\mathbf{R})$-conjugacy class X of such maps.

Example 4.1. In our earlier considerations with $G = \mathrm{GL}_2$, we saw that the maps $\mathbf{C}^\times \to \mathrm{GL}_2(\mathbf{R})$ arising from $\mathbf{R}$-algebra embeddings $\mathbf{C} \to \mathrm{Mat}_2(\mathbf{R})$ (i.e., $\mathbf{C}$-linear structures on $\mathbf{R}^2$) constitute such a $G(\mathbf{R})$-conjugacy class when $G = \mathrm{GL}_2$.

For any $h : \mathbf{C}^\times \to G(\mathbf{R})$, its centralizer $Z_h \subseteq G(\mathbf{R})$ under the $G(\mathbf{R})$-action (i.e., the group of $g \in G(\mathbf{R})$ such that $gh(z)g^{-1} = h(z)$ for all $z \in \mathbf{C}^\times$) is a closed subgroup. (In fact, by Remark 3.3, it is the group of $\mathbf{R}$-points of a canonically determined Zariski-closed $\mathbf{R}$-subgroup of G.) It is clear that $Z_{g_0.h} = g_0 Z_h g_0^{-1}$ for any $g_0 \in G(\mathbf{R})$, and the identification of sets

$$\xi_h : G(\mathbf{R})/Z_h \simeq X$$

defined by $g \mapsto g.h$ satisfies $g_0.\xi_h = \xi_{g_0.h} \circ c_{g_0}$, where $c_{g_0} : G(\mathbf{R})/Z_h \simeq G(\mathbf{R})/Z_{g_0.h}$ is defined by $g \mapsto g_0 g g_0^{-1}$. Thus, since c_{g_0} is a real-analytic isomorphism (using the natural real-analytic structure on the quotient of any Lie group modulo a closed subgroup), it follows that via any choice of h the bijection ξ_h equips X with a structure of real-analytic manifold that is independent of the choice of h.

The elegance of Deligne's emphasis on the homogeneous space X for $G(\mathbf{R})$ rather than on its concrete description $G(\mathbf{R})/Z_h$ upon choosing $h \in X$ is that this avoids the non-canonical choice of base point (and thereby leads to cleaner arguments in practice).

Example 4.2. Consider $G = \mathrm{GL}_2$ and X as in Example 4.1. Then $Z_h = h(\mathbf{C}^\times)$, so by using the base point h we get the description $X \simeq G(\mathbf{R})/h(\mathbf{C}^\times)$. Consider the *canonical* map $X \to \mathbf{C} - \mathbf{R}$ that carries $h \in X$ to the unique $\tau(h) \in \mathbf{C} - \mathbf{R}$ such that $e_1 = h(\tau(h))e_2$. (Note that h corresponds to a complex structure on $\mathbf{R}^2$, relative to which any ordered basis admits a unique "ratio" in $\mathbf{C} - \mathbf{R}$. This is why $\tau(h)$ exists and is uniquely determined.) Just as X has no canonical base point, neither does $\mathbf{C} - \mathbf{R}$. The "coordinatized" appearance of X as $\mathbf{C} - \mathbf{R}$ is related to the fact that our algebraic group G has also been "coordinatized" as $\mathrm{GL}_2 = \mathrm{GL}(W)$ for $W = \mathbf{R}^2$.

More conceptually, we should really view $\mathbf{C} - \mathbf{R}$ as $\mathbf{P}^1(\mathbf{C}) - \mathbf{P}^1(\mathbf{R}) = \mathbf{P}(W_{\mathbf{C}}) - \mathbf{P}(W)$ where $\mathbf{P}(W)$ is taken in the sense of Grothendieck and *not* in the classical sense: it is the moduli space for nonzero linear functionals on W (i.e., lines in the dual space W^*) rather than the moduli space of lines in W (and similarly for $W_{\mathbf{C}} = \mathbf{C}^2$ in place of $W = \mathbf{R}^2$). The reason for using Grothendieck's convention (apart from the fact that Grothendieck is always right) is that there is an elegant *coordinate-free* description of the canonical map $X \to \mathbf{P}(W_{\mathbf{C}}) - \mathbf{P}(W)$ defined above in "coordinates" via $h \mapsto \tau(h)$. This goes as follows.

For any $h \in X$, the $\mathbf{C}$-linear automorphism $1 \otimes h(\tau(h))$ of $\mathbf{C} \otimes_{\mathbf{R}} W = W_{\mathbf{C}}$ acts with distinct eigenvalues $\tau(h)$ and $\overline{\tau(h)}$ since h comes from an $\mathbf{R}$-algebra embedding $\mathbf{C} \to \mathrm{Mat}_2(\mathbf{R}) = \mathrm{End}(W)$ (and Cayley-Hamilton provides a monic quadratic over $\mathbf{R}$ that is satisfied by $h(\tau(h))$, so that must be the irreducible quadratic minimal polynomial of $\tau(h)$ over $\mathbf{R}$). Under the corresponding $\mathbf{C}$-linear eigenline decomposition

$$W_{\mathbf{C}} = W_{\tau(h)} \oplus W_{\overline{\tau(h)}},$$

the $\mathbf{C}$-line $W_{\overline{\tau(h)}}$ has annihilator $W_{\overline{\tau(h)}}^{\perp}$ that is a line in $W_{\mathbf{C}}^*$. We claim that via the identification $W_{\mathbf{C}} = \mathbf{C}^2$, the point in $\mathbf{P}(W_{\mathbf{C}}) = \mathbf{P}^1(\mathbf{C})$ corresponding to the line $W_{\overline{\tau(h)}}^{\perp}$ in $W_{\mathbf{C}}^*$ is *exactly* $\tau(h)$! Indeed, under

Grothendieck's convention a point $[\tau, 1] \in \mathbf{P}^1(\mathbf{C})$ corresponds to the homothety class of the linear form $\tau e_1^* + e_2^*$ whose kernel line is the span of $e_1 - \tau e_2$, so it suffices to show that the visibly nonzero $1 \otimes e_1 - \tau(h) \otimes e_2 \in \mathbf{C} \otimes_{\mathbf{R}} W$ is a $\overline{\tau(h)}$-eigenvector for $1 \otimes h(\tau(h))$. This is a straightforward calculation: since $h(\tau(h))e_2 = e_1$ by definition of $\tau(h)$,

$$\begin{aligned}(1 \otimes h(\tau(h)))(1 \otimes e_1 - \tau(h) \otimes e_2) &= 1 \otimes h(\tau(h))(e_1) - \tau(h) \otimes h(\tau(h))(e_2) \\ &= 1 \otimes h(\tau(h))(h(\tau(h))(e_2)) - \tau(h) \otimes e_1,\end{aligned}$$

and $h(\tau(h))^2 = h(\tau(h)^2) = h(-u\tau(h) - v) = -uh(\tau(h)) - v$ where $t^2 + ut + v \in \mathbf{R}[t]$ is the minimal polynomial of $\tau(h)$ over $\mathbf{R}$. Thus,

$$\begin{aligned}1 \otimes h(\tau(h))^2(e_2) - \tau(h) \otimes e_1 = 1 \otimes (-uh(\tau(h)) - v)(e_2) - \tau(h) \otimes e_1 &= 1 \otimes (-ue_1 - ve_2) - \tau(h) \otimes e_1 \\ &= (-u - \tau(h)) \otimes e_1 - v \otimes e_2,\end{aligned}$$

and $-u - \tau(h) = \overline{\tau(h)}$ and $v = \mathrm{N}_{\mathbf{C}/\mathbf{R}}(\tau(h)) = \overline{\tau(h)}\tau(h)$. Putting it all together,

$$(1 \otimes h(\tau(h)))(1 \otimes e_1 - \tau(h) \otimes e_2) = \overline{\tau(h)}(1 \otimes e_1 - \tau(h) \otimes e_2),$$

as required.

Example 4.3. Pushing the preceding example a bit further, it is natural to ask how the canonical left $\mathrm{GL}_2(\mathbf{R})$-action on X translates over to an action on $\mathbf{C} - \mathbf{R}$ via the canonical identification of X with $\mathbf{C} - \mathbf{R}$. This translation turns out *not* to be the classical linear fractional transformation action of $\mathrm{GL}_2(\mathbf{R})$ on $\mathbf{C} - \mathbf{R}$. Instead, we claim that it is the composition of that action with the transpose-inverse automorphism of $\mathrm{GL}_2(\mathbf{R})$.

The intervention of transpose-inverse in this claim is not a surprise, in view of the fact that we have seen the appropriateness of interpreting projective lines in the sense of Grothendieck: the natural *left* action of $\mathrm{GL}(W)$ on the Grothendieck projective space $\mathbf{P}(W)$ of lines in W^* goes via inner composition through inversion, which is to say that it rests on the isomorphism $\mathrm{GL}(W) \simeq \mathrm{GL}(W^*)$ defined by $g \mapsto (g^{-1})^*$. This isomorphism is transpose-inverse in the language of matrices for $W = \mathbf{R}^2$.

To verify our claim, we fix a base point $h_0 \in X$ and will compute the composite map

$$\mathrm{GL}_2(\mathbf{R})/h_0(\mathbf{C}^\times) \simeq X \simeq \mathbf{C} - \mathbf{R}$$

in terms of $\tau(h_0) \in \mathbf{C} - \mathbf{R}$. Any $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{R})$ is carried to $g.h_0 \in X$, which in turn goes to the point $\tau(g.h_0) \in \mathbf{C} - \mathbf{R}$ characterized by the property

$$e_1 = (g.h_0)(\tau(g.h_0))e_2 = (g \circ h_0(\tau(g.h_0)) \circ g^{-1})(e_2),$$

so $g^{-1}e_1 = h_0(\tau(g.h_0))(g^{-1}e_2)$. Scaling through by $\det g \in \mathbf{R}^\times$ is harmless, so this says

$$de_1 - ce_2 = h_0(\tau(g.h_0))(-be_1 + ae_2).$$

But $e_1 = h_0(\tau(h_0))e_2$ and h_0 comes from an $\mathbf{R}$-algebra map $\mathbf{C} \to \mathrm{Mat}_2(\mathbf{R})$, so $de_1 - ce_2 = h_0(d\tau(h_0) - c)e_2$ and $-be_1 + ae_2 = h_0(-bh_0(\tau(h_0)) + a)e_2$. Hence,

$$h_0(d\tau(h_0) - c)e_2 = h_0(\tau(g.h_0)(-bh_0(\tau(h_0)) + a))e_2$$

and thus the ratio $\tau(g.h_0)(-bh_0(\tau(h_0)) + a)/(d\tau(h_0) - c) \in \mathbf{C}^\times$ fixes e_2 under the $\mathbf{C}$-linear structure defined by h_0. It follows that this ratio is 1, so

$$\tau(g.h_0) = \frac{d\tau(h_0) - c}{-bh_0(\tau(h_0)) + a} = [(g^{-1})^t](\tau(h_0))$$

in $\mathbf{C}^\times$. In other words, the composite identification $\mathrm{GL}_2(\mathbf{R})/h_0(\mathbf{C}^\times) \simeq \mathbf{C} - \mathbf{R}$ intertwines the left translation action by $\mathrm{GL}_2(\mathbf{R})$ on the source and the composition of transpose-inverse with the classical linear fractional action on the target, as desired.

Returning to our consideration of general G, for any $h \in X$ and algebraic representation $\rho : G \to \mathrm{GL}(W)$ (i.e., homomorphism of $\mathbf{R}$-groups, with W a finite-dimensional $\mathbf{R}$-vector space), composing ρ on $\mathbf{R}$-points with $h : \mathbf{C}^\times \to G(\mathbf{R})$ defines an algebraic homomorphism $\rho \circ h$ from $\mathbf{C}^\times$ to $\mathrm{GL}(W)(\mathbf{R}) = \mathrm{Aut}(W)$. Thus, the adjoint representation of $\mathrm{GL}(W)$ on $\mathrm{Lie}(\mathrm{GL}(W)) = \mathrm{End}(W)$ defines an $\mathbf{R}$-linear action of $\mathbf{C}^\times$ on $\mathrm{End}(W)$. Explicitly, this action is $z.T = \rho(h(z))T\rho(h(z))^{-1}$ for $z \in \mathbf{C}^\times$. For example, if $G = \mathrm{GL}_2$ and we take ρ to be the equality of G with $\mathrm{GL}(\mathbf{R}^2)$ then we get the familiar linear action of $\mathbf{C}^\times$ on $\mathrm{Mat}_2(\mathbf{R})$ via $z.T = h(z)Th(z)^{-1}$, which makes $\mathbf{R}^\times$ act trivially.

Note that in general the $\mathbf{C}^\times$-action on $\mathrm{End}(W)$ satisfies an algebraicity property: the action map $\mathbf{C}^\times \to \mathrm{Aut}(\mathrm{End}(W))$ has restriction to $\mathbf{R}^\times$ that is *algebraic* in the sense of Definition 3.2 (i.e., it arises from an $\mathbf{R}$-group map $\mathrm{GL}_1 \to \mathrm{GL}(\mathrm{End}(W))$), due to the algebraicity hypothesis on h and the fact that the formation of the adjoint representation of a Lie group is compatible with an analogous operation for algebraic groups. (That is, if H is a linear algebraic $\mathbf{R}$-group then there is a natural $\mathbf{R}$-group map $\mathrm{Ad}_H : H \to \mathrm{GL}(\mathrm{Lie}(H))$ which on $\mathbf{R}$-points recovers the usual adjoint representation of $H(\mathbf{R})$ via the natural identification of $\mathrm{Lie}(H)$ and $\mathrm{Lie}(H(\mathbf{R}))$.)

Linear actions of $\mathbf{C}^\times$ on finite-dimensional $\mathbf{R}$-vector spaces U such that the action map $\mathbf{C}^\times \to \mathrm{Aut}(U)$ is algebraic will be called *algebraic* actions of $\mathbf{C}^\times$ on U. (Using Remark 3.3, this is precisely the condition of arising from a necessarily unique $\mathbf{R}$-group homomorphism $\mathbf{S} \to \mathrm{GL}(U)$.) The basic example of interest to us are the actions on $\mathrm{End}(W)$ as above. Such actions have an interesting interpretation, but to formulate the interpretation we need a definition:

Definition 4.4. Let U be a finite-dimensional $\mathbf{R}$-vector space. A (real) *Hodge structure* on U is a $\mathbf{C}$-linear direct sum decomposition $U_{\mathbf{C}} = \oplus_{p,q} V^{p,q}$ such that the natural conjugate-linear action $c \otimes u \mapsto \overline{c} \otimes u$ of complex conjugation on $U_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} U$ swaps $V^{p,q}$ and $V^{q,p}$. The decreasing chain of subspaces

$$F^p = \bigoplus_{p' \geq p,\ q' \in \mathbf{Z}} V^{p',q'}$$

is called the *Hodge filtration* of $U_{\mathbf{C}}$, and if the (p,q) for which $V^{p,q} \neq 0$ all satisfy $p+q=n$ then we say that the Hodge structure is *pure of weight* n.

Note that in general $F^p \cap \overline{F}^q = \oplus_{p' \geq p, q' \geq q} V^{p',q'}$, so if the Hodge structure is pure of weight n then $F^p \cap \overline{F}^q = V^{p,q}$ whenever $p+q=n$ (so in the pure case the Hodge structure can be recovered from the Hodge filtration by means of the complex conjugation action on $U_{\mathbf{C}}$). Also, it is generally harmless to restrict attention to pure Hodge structures, since if U is equipped with a Hodge structure then for any $n \in \mathbf{Z}$ the $\mathbf{C}$-subspaces $\oplus_{p+q=n} V^{p,q}$ in $U_{\mathbf{C}}$ are stable under the complex conjugation action $c \otimes u \mapsto \overline{c} \otimes u$ (which swaps $V^{p,q}$ and $V^{q,p}$ by hypothesis) and hence descend to $\mathbf{R}$-subspaces $U_n \subseteq U$ such that $\oplus U_n \simeq U$ (as scalar extension to $\mathbf{C}$ recovers the Hodge decomposition of $U_{\mathbf{C}}$, with the $V^{p,q}$ collected according to the value of $p+q$). Each U_n is equipped with a pure Hodge structure of weight n, so we can study these separately.

Example 4.5. Let's see how Hodge structures of a very special type encode the data of a $\mathbf{C}$-linear structure on a finite-dimensional $\mathbf{R}$-vector space U of even dimension. Suppose that U is equipped with a $\mathbf{C}$-linear structure, and let $\overline{U}$ denote the conjugate $\mathbf{C}$-linear structure (i.e., compose the chosen $\mathbf{R}$-algebra map $\mathbf{C} \to \mathrm{End}(U)$ with complex conjugation on $\mathbf{C}$). In other words, $\overline{U} = \overline{\mathbf{C}} \otimes_{\overline{z}, \mathbf{C}} U$. The natural map

$$\xi : \mathbf{C} \otimes_{\mathbf{R}} U \to U \oplus \overline{U}$$

defined by $c \otimes u \mapsto (cu, \overline{c}u)$ (the second component using the initial $\mathbf{C}$-linear structure, so it is also c-multiplication relative to the conjugate $\mathbf{C}$-linear structure) is $\mathbf{C}$-*linear*, and we claim it is an isomorphism. Indeed, observe that the natural map of $\mathbf{R}$-algebras $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \simeq \mathbf{C} \times \mathbf{C}$ defined by $a \otimes b \mapsto (ab, a\overline{b})$ is an isomorphism (as for any finite Galois extension of fields) and it is $\mathbf{C}$-linear when using the diagonal structure on the target and the left tensor structure on the source, as well as $\mathbf{C}$-linear when using the twisted diagonal structure $c.(a,b) = (ac, b\overline{c})$ on the target and the right tensor structure on the source. Thus, we get an isomorphism

$$\mathbf{C} \otimes_{\mathbf{R}} U = (\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}) \otimes_{\mathbf{C}} U \simeq (\mathbf{C} \times \mathbf{C}) \otimes_{\mathbf{C}} U \simeq U \oplus \overline{U}$$

which is readily checked to coincide with the map ξ. Observe from the definitions that the natural complex conjugation action on $\mathbf{C} \otimes_{\mathbf{R}} U$ via $c \otimes u \mapsto \overline{c} \otimes u$ has the effect of swapping the direct factor subspaces U and $\overline{U}$.

Turning this around, suppose we're given a $\mathbf{C}$-linear decomposition $\mathbf{C} \otimes_{\mathbf{R}} U = W \oplus W'$ such that complex conjugation on $\mathbf{C} \otimes_{\mathbf{R}} U$ swaps W and W' (i.e., $\overline{W} = W'$ and $\overline{W}' = W$). This forces $\dim_{\mathbf{R}} U = 2 \dim_{\mathbf{C}} W = \dim_{\mathbf{R}} W$ to be even, and the natural map $U \to (\mathbf{C} \otimes_{\mathbf{R}} U)/W' = W$ is an $\mathbf{R}$-linear isomorphism since the dimensions agree and the map is injective (as $U \cap W'$ must vanish, since U is the fixed locus of complex conjugation on $\mathbf{C} \otimes_{\mathbf{R}} U$ yet $\overline{W}' = W$ meets W' in 0). This $\mathbf{R}$-linear isomorphism puts a $\mathbf{C}$-linear structure on U and reverses the preceding procedure.

Summarizing, to put a complex structure on U is the same as to given a $\mathbf{C}$-linear decomposition $\mathbf{C} \otimes_{\mathbf{R}} U = W \oplus \overline{W}$. In such cases the action of $\mathbf{C}^{\times}$ on U under the complex structure goes over to the actions $z.w = zw$ and $z.\overline{w} = \overline{zw}$ on W and $\overline{W}$ respectively, so this is a Hodge structure on U of type $\{(-1,0),(0,-1)\}$. In this way, Hodge structures are a vast generalization of complex structures.

Example 4.6. The ur-example of a Hodge structure that is pure of weight n occurs for $U_n = \mathrm{H}^n(Y(\mathbf{C}), \mathbf{R})$ with a smooth proper variety Y over $\mathbf{C}$. The complex-analytic structure on $Y(\mathbf{C})$ defines a decreasing *Hodge filtration* F^p (consisting of de Rham cohomology classes represented by closed $\mathbf{C}$-valued C^{∞} differential forms whose local expression in terms of $\mathrm{d}z_j$'s and $\mathrm{d}\overline{z}_k$'s only involves wedge products with at least p of the $\mathrm{d}z_j$'s; e.g., $F^0 = (U_n)_{\mathbf{C}}$, F^n contains the space $\mathrm{H}^0(Y(\mathbf{C}), \Omega^n_{Y(\mathbf{C})})$ of global holomorphic n-forms, and $F^{n+1} = 0$). The cohomology algebra $\oplus_{n \geq 0} U_n = \mathrm{H}^*(Y(\mathbf{C}), \mathbf{R})$ naturally inherits a Hodge structure for which U_n is the associated subspace that is pure of weight n.

For instance, if $Y = E$ is an elliptic curve over $\mathbf{C}$ and we take $n = 1$ then $F^1 = \mathrm{H}^0(E, \Omega^1_E)$ is the space of global 1-forms on E and it is the kernel of the natural map $F^0 = (U_n)_{\mathbf{C}} = \mathrm{H}^1(E(\mathbf{C}), \mathbf{C}) \to \mathrm{H}^1(E(\mathbf{C}), \mathscr{O}_{E(\mathbf{C})}) = \mathrm{H}^1(E, \mathscr{O}_E)$ which turns out to be surjective.

It is a deep theorem that for U_n as above, $V^{p,q} := F^p \cap \overline{F}^q$ defines a Hodge structure on U_n that is pure of weight n (e.g., $V^{p,q} =$ whenever $p + q \neq n$). Traditionally this is proved by exhibiting another construction of $V^{p,q}$ in terms of "harmonic forms" (relative to a choice of Kähler metric, say when Y is projective), but the final output of the construction turns out to be metric-independent. An alternative purely algebraic proof by Deligne–Illusie avoids reference to a metric.

Example 4.7. The notion of a Hodge structure over $\mathbf{Z}$ or $\mathbf{Q}$ is defined in a similar manner using the scalar extension to $\mathbf{R}$: a finitely generated module U over $\mathbf{Z}$ or $\mathbf{Q}$ such that $U_{\mathbf{R}}$ is equipped with a real Hodge structure in the sense defined above. The interest in this notion is due to the examples of integral and rational cohomology of smooth projective varieties over $\mathbf{C}$. Exactly as we define operations of tensor product and dual in representation theory, there are natural notions of morphism, tensor product, and dual among Hodge structures (over $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$, requiring torsion-freeness for duality in the $\mathbf{Z}$-case), and these are compatible with cohomological operations (such as the Künneth formula for cohomology of a product variety). For example, if U and U' are real Hodge structures and $V = U_{\mathbf{C}}$ and $V' = U'_{\mathbf{C}}$ then by definition the Hodge structures on $U \otimes U'$ and U^* satisfy $(V^*)^{p,q} = V^{-p,-q}$ and

$$(V \otimes_{\mathbf{C}} V')^{p,q} = \bigoplus_{a+a'=p, b+b'=q} V^{a,b} \otimes_{\mathbf{C}} V'^{a',b'}.$$

Thus, the Hodge filtrations are $F^p(V^*) = (F^{1-p})^{\perp}$ and $F^p(V \otimes V') = \sum_{a+a'=p} F^a \otimes F'^{a'}$.

An important example of a Hodge structure over $\mathbf{Z}$ (and hence by scalar extension, over $\mathbf{Q}$ and $\mathbf{R}$) is the *Tate object* $\mathbf{Z}(1) := \ker(\exp : \mathbf{C} \to \mathbf{C}^{\times})$. We make this pure of weight -2 by declaring its scalar extension $V = \mathbf{C}(1)$ to be $V^{-1,-1}$. Thus, the Hodge filtration in this case is $F^p = V$ for $p < 0$ and $F^p = 0$ for $p \geq 0$. Passing to duals and tensor powers, we likewise get a Hodge structure on the nth tensor power $\mathbf{Z}(n)$ (and $\mathbf{Q}(n)$ and $\mathbf{R}(n)$) for all $n \in \mathbf{Z}$ via declaring $\mathbf{C}(n)^{-n,-n} = \mathbf{C}(n)$ (i.e., $V^{p,q} = 0$ for all $(p,q) \neq (-n,-n)$), so this is pure of weight $-2n$. For instance, if Y is a smooth connected proper variety over $\mathbf{C}$ with dimension N then $\mathrm{H}^{2N}(Y(\mathbf{C}), \mathbf{Z}) \simeq \mathbf{Z}(-N)$ via the i-independent integration operation $(2\pi i)^{-N} \int_{Y(\mathbf{C}),i}$, and this is an isomorphism of pure Hodge structures of weight $2N$ because the Hodge structure on the top cohomology with **C**-coefficients has its unique nonzero term in bidegree (N, N) (as every C^∞-form on an N-dimensional complex manifold involves all N of the $\mathrm{d}z_j$'s and all N of the $\mathrm{d}\overline{z}_k$'s)

The n-fold *Tate twist* $U(n)$ is defined to be $U \otimes \mathbf{Z}(n)$ in the **Z**-case, and similarly in the **Q**-case and **R**-case. Concretely, if $V = U_{\mathbf{C}}$ then $U(n)_{\mathbf{C}} = V \otimes_{\mathbf{C}} \mathbf{C}(n) = V$ via the canonical identification $\mathbf{C}(n) \simeq \mathbf{C}$ defined by *multiplication* within **C**. (The canonical basis of $\mathbf{C}(n)$ is $(2\pi i)^{-n} \otimes (2\pi i)^{\otimes n}$ for either $i = \sqrt{-1} \in \mathbf{C}$.) In this way, we have $U(n)^{p,q}_{\mathbf{C}} = V^{p+n,q+n}$ for all (p, q) and any $n \in \mathbf{Z}$. Hence, the Hodge filtration on $U(n)_{\mathbf{C}}$ has pth stage F^{p+n}.

Example 4.8. For Hodge structures U and U' (with torsion-freeness in the **Z**-case), since $\mathrm{Hom}(U, U') = U' \otimes U^*$ we have a notion of "internal Hom" for Hodge structures. That is, for Hodge structures U and U', the space $\mathrm{Hom}(U, U')$ is equipped with a naturally associated Hodge structure. For later applications, let's work out what the Hodge filtration means in terms of Hom's (without reference to the isomorphism $\mathrm{Hom}(U, U') = U' \otimes U^*$).

Letting $V = U_{\mathbf{C}}$ and $V' = U'_{\mathbf{C}}$, so $\mathrm{Hom}(U, U')_{\mathbf{C}} = \mathrm{Hom}(V, V')$, we claim that

$$F^p(\mathrm{Hom}(V, V')) = \{T \in \mathrm{Hom}(V, V') \,|\, T(F^a) \subseteq F'^{p+a} \text{ for all } a \in \mathbf{Z}\};$$

for instance, $F^0(\mathrm{Hom}(V, V'))$ consists of **C**-linear maps $V \to V'$ which respect the Hodge filtration (but perhaps not the Hodge decomposition!) and in general $F^p(\mathrm{Hom}(V, V')) = F^0(\mathrm{Hom}(U, U'(p))_{\mathbf{C}})$ via the canonical identification of V' with $V'(p)$. Indeed, note that

$$F^p(\mathrm{Hom}(U, U')_{\mathbf{C}}) = \sum_{a+a'=p} (F^{1-a})^\perp \otimes F'^{a'} = \sum_a \mathrm{Hom}(V/F^{1-a}, F'^{p-a}),$$

and $V/F^{1-a} = \oplus_{p' \leq -a, q' \in \mathbf{Z}} V^{p',q'}$. Since $V^{p',q'}$ occurs in this latter direct sum precisely when $a \leq -p'$, and F'^{p-a} is "increasing" with respect to a, a map T lies in the pth filtered piece precisely when it carries each $V^{p',q'}$ into $F'^{p+p'}$ for all (p', q'). This says exactly that T carries F^a into F'^{p+a} for all a.

Pure Hodge structures can be encoded in terms of the following nifty piece of linear algebra:

Proposition 4.9. *For a finite-dimensional* **R***-vector space* U*, to equip* U *with a Hodge structure is equivalent to specifying an algebraic linear action* $h : \mathbf{C}^\times \to \mathrm{Aut}(U)$*. Under this equivalence, the subspace* $V^{p,q} \subseteq U_{\mathbf{C}}$ *consists of those* $v \in U_{\mathbf{C}}$ *such that* $(1 \otimes h(z))(v) = z^{-p}\overline{z}^{-q} v$ *for all* $z \in \mathbf{C}^\times$*, and it is pure of weight* n *if and only if* $h(x) = x^{-n}$ *for all* $x \in \mathbf{R}^\times$*.*

Remark 4.10. For the reader who knows a bit about linear algebraic groups, this result says that linear representations of the Deligne torus $\mathbf{S} = \mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathrm{GL}_1)$ on a finite-dimensional vector space U over **R** are (functorially) "the same" as Hodge structures on U.

Before we prove Proposition 4.9, let's explain a reason for the sign conventions in the proposition (e.g., using $z^{-p}\overline{z}^{-q}$ rather than $z^p\overline{z}^q$). In the examples arising from moduli of elliptic curves (and abelian varieties), the relevant Hodge structures occur on the degree-1 *homology* of elliptic curves (and abelian varieties). This is dual to degree-1 cohomology. Under any reasonable notion of "weight" in representation theory (such as for S^1-actions on complex vector spaces), passing to the dual vector space should negate the weight. One reason to declare degree-n cohomology to have weight n rather than weight $-n$ is that the action of geometric Frobenius on the degree-n étale cohomology of smooth proper varieties over finite fields has weight n in the sense of Weil numbers. (This "consistency" makes the Artin comparison isomorphism between ℓ-adic topological and étale cohomologies be weight-compatible.)

Here is the proof of the proposition.

Proof. The purity criterion at the end of the proposition follows from the rest, since the restriction of $z^{-p}\overline{z}^{-q}$ to $\mathbf{R}^\times$ is $x^{-(p+q)}$. It is a general fact in the theory of algebraic tori (see Proposition 8.2(d) in Borel's book "Linear Algebraic Groups") that for any map of **R**-groups $\mathrm{GL}_1 \to \mathrm{GL}(U)$ there is a unique GL_1-stable linear decomposition $U = \oplus_n U_n$ such that GL_1 acts on U_n through scaling by the nth-power (i.e., $x.u = x^n u$ for $x \in \mathrm{GL}_1$ and $u \in U_n$). Since the $h(\mathbf{C}^\times)$-action commutes with the $h(\mathbf{R}^\times)$-action, it preserves the U_n's. The only endomorphisms of the **R**-group GL_1 are the power maps $x \mapsto x^n$, so in view of (i) the algebraicity requirement on $h|_{\mathbf{R}^\times}$, (ii) the proposed recipe in the proposition, and (iii) the decomposition of any Hodge structure into a direct sum of pure Hodge structures, we can pass separately to the U_n's to reduce to the case that U is pure of some weight n.

Now our task is to identify Hodge structures on U that are pure of weight n and $\mathbf{R}$-linear actions of $\mathbf{C}^\times$ on U such that $\mathbf{R}^\times$ acts through scaling by the $-n$th-power map. In one direction, given such an action we consider the weight space decomposition for the resulting $\mathbf{C}$-linear S^1-action on $V = U_{\mathbf{C}}$, say $V = \oplus_r V_r$ where $z \in S^1$ acts on V_r through scaling by z^r. The action of $-1 \in S^1$ is via multiplication by $(-1)^n$, so $V_r \neq 0$ only when $r \equiv n \bmod 2$. Define $V^{p,n-p} = V_{n-2p}$ and $V^{p,q} = 0$ otherwise. The action of $z \in S^1$ on $V^{p,n-p}$ is scaling by $z^{n-2p} = z^{-p}z^{n-p} = z^{-p}\overline{z}^{-(n-p)}$ since $\overline{z} = 1/z$ for $z \in S^1$. The same formula works for $z = x \in \mathbf{R}^\times$ since $\overline{z} = x$ and we have arranged that x acts on U through scaling by x^{-n}. This yields the desired the procedure in one direction, and it goes in reverse. ■

Example 4.11. For our motivating example associated to GL_2, with $U = \mathbf{R}^2$, the maps $h \in X$ correspond to Hodge structures on $\mathbf{R}^2$ of weight -1 (since $h(x) = x = x^{-(-1)}$ for $x \in \mathbf{R}^\times$).

5. Shimura data

Let G be a connected linear algebraic group over $\mathbf{R}$, and X a $G(\mathbf{R})$-conjugacy class of algebraic homomorphisms $h : \mathbf{C}^\times \to G(\mathbf{R})$. In the classical setting of Example 4.2 (and Example 4.3) with $G = \mathrm{GL}_2$ and the specific X considered there, we saw that X is identified with the moduli space for variations of complex structure on $\mathbf{R}^2$. In general, we wish to axiomatize situations for which X should serve as a "moduli space" for Hodge structures equipped with G-action (and in particular, the real-analytic manifold X should have a canonical complex-analytic structure).

The points h serve as abstract encodings of Hodge structures, in the sense that *any* algebraic representation $\rho : G \to \mathrm{GL}(U)$ over $\mathbf{R}$ induces a collection of algebraic homomorphisms $\rho_{\mathbf{R}} \circ h : \mathbf{C}^\times \to \mathrm{Aut}(U)$ (where $\rho_{\mathbf{R}}$ denotes ρ restricted to $\mathbf{R}$-points) and hence a "family" of Hodge structures on U (via Proposition 4.9) parameterized by X. There is a particularly special choice for (ρ, U), namely the adjoint representation $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ for $\mathfrak{g} = \mathrm{Lie}(G) = \mathrm{Lie}(G(\mathbf{R}))$. One of the key principles underlying Deligne's viewpoint is that the validity of certain Hodge-theoretic conditions on the Hodge structures arising from *every* (ρ, U) should be expressed entirely in terms of the pair (G, X) and the Hodge structure on $\mathfrak{g}$ arising from $\rho = \mathrm{Ad}_G$.

For any (ρ, U), the $\mathbf{R}$-linear weight decomposition

$$\bigoplus_{n \in \mathbf{Z}} U_{n,h}$$

on U arising from each $h \in X$ is (by definition) the decomposition arising from the GL_1-action $\rho_{\mathbf{R}} \circ h|_{\mathbf{R}^\times}$ on U over $\mathbf{R}$. (Concretely, $U_{n,h}$ consists of those $u \in U$ such that $\rho(h(x))(u) = x^n u$ for all $x \in \mathbf{R}^\times$.) We first wish to formulate a condition on (G, X) that is equivalent to the property that for *any* (ρ, U) the subspaces $U_{n,h} \subseteq U$ are the same for all $h \in X$. This will allow us to pass to the individual $U_{n,h}$'s and thereby focus our attention on the case of pure Hodge structures "parameterized" by X. The desired condition comes from:

Lemma 5.1. *In order that the subspaces $U_{n,h} \subseteq U$ be independent of $h \in X$ for every $\rho : G \to \mathrm{GL}(U)$, it is necessary and sufficient that the algebraic map $w_h : \mathrm{GL}_1 \to G$ corresponding to $h|_{\mathbf{R}^\times} : \mathbf{R}^\times \to G(\mathbf{R})$ is independent of h. This happens if and only if w_h has image in the center of G for some* (*equivalently, all*) *$h \in X$.*

Under the conditions in this lemma, the resulting common map $w : \mathrm{GL}_1 \to Z_G$ into the center of G is called the *weight homomorphism* attached to (G, X).

Proof. For any $g \in G(\mathbf{R})$, $\rho(w_{g.h}(x)) = \rho(g)\rho(w_h(x))\rho(g)^{-1}$ for all $x \in \mathbf{R}^\times$. Since $\rho \circ w_h$ encodes exactly the information of the decomposition $\oplus_n U_{n,h}$ of U (by the general story of algebraic representations of GL_1, as noted near the beginning of the proof of Proposition 4.9), and the set X is a $G(\mathbf{R})$-conjugacy class of h's, it is necessary and sufficient that $\rho \circ w_h(x)$ is centralized by $\rho(g)$ for all $g \in G(\mathbf{R})$, all $x \in \mathbf{R}^\times$, and all $\rho : G \to \mathrm{GL}(U)$. Taking ρ that is faithful (as we may always do: every linear algebraic group over a field arises as a closed subgroup of some GL_n), it is necessary and sufficient that $gw_h(x)g^{-1} = w_h(x)$ for all $g \in G(\mathbf{R})$ and $x \in \mathbf{R}^\times$. But $G(\mathbf{R})$ is Zariski-dense in G for any connected linear algebraic group G over $\mathbf{R}$ (this is valid for any field of characteristic 0 in place of $\mathbf{R}$, and as such follows from general facts in the theory of linear algebraic groups). Hence, the centralizer of $G(\mathbf{R})$ in G is the center Z_G of G in the sense of algebraic groups, so it is equivalent to say that $w_h(x) \in Z_G(\mathbf{R})$ for all $x \in \mathbf{R}^\times$.

But $\mathbf{R}^\times$ is Zariski-dense in GL_1 over $\mathbf{R}$, so it is equivalent to say that $w_h : \mathrm{GL}_1 \to G$ lands in Z_G. Since $w_{g.h}(x) = gw_h(x)g^{-1}$ for $g \in G(\mathbf{R})$ and $x \in \mathbf{R}^\times$, we deduce the equivalence with the condition that w_h is independent of $h \in X$. ■

Let's phrase the centrality of $w_h : \mathrm{GL}_1 \to G$ in another way. Since G is connected as an algebraic group, an $\mathbf{R}$-automorphism of G is the identity if and only if it induces the identity on $\mathfrak{g} = \mathrm{Lie}(G)$. In particular, $h(x)$ centralizes G if and only if $\mathrm{Ad}(h(x)) \in \mathrm{Aut}(\mathfrak{g})$ is trivial, or equivalently the algebraic action $\mathbf{C}^\times \to \mathrm{Aut}(\mathfrak{g})$ makes $\mathbf{R}^\times$ act with weight 0.

By Proposition 4.9 and the discussion preceding Definition 4.4, for any $h \in X$ the composite algebraic homomorphism

$$\mathbf{C}^\times \xrightarrow{h} G(\mathbf{R}) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g})$$

corresponds to a Hodge structure on $\mathfrak{g}$. Lemma 5.1 shows that this has weight 0 (i.e., trivial $\mathbf{R}^\times$-action) if and only if for *every* algebraic representation $\rho : G \to \mathrm{GL}(U)$ the Hodge structures on U arising from the $\rho_{\mathbf{R}} \circ h : \mathbf{C}^\times \to \mathrm{Aut}(U)$ all yield the same weight decomposition on U. This motivates the importance of:

Axiom 0. For the pair (G, X), the induced Hodge structure on $\mathfrak{g} = \mathrm{Lie}(G)$ is pure of weight 0.

This is labeled as Axiom 0 because it will be a consequence of a later Axiom I.

Example 5.2. In our favorite example with $G = \mathrm{GL}_2 = \mathrm{GL}(\mathbf{R}^2)$, Axiom 0 is satisfied: it amounts to the fact that $h(\mathbf{R}^\times)$ is contained in the center of $G(\mathbf{R})$ for some (equivalently, all) $h \in X$. Beware that this is *not* to be confused with the fact that the Hodge structures on $\mathbf{R}^2$ arising from the elements $h \in X$ do *not* have weight 0 and in fact are pure of weight -1 (Example 4.11)!

We can push this a bit further. Since the weight is 0, the Hodge decomposition on $\mathfrak{g}$ associated to $\mathrm{Ad}_{G(\mathbf{R})} \circ h$ has the form $\{V_h^{p,-p}\}$ for various subspaces $V_h^{p,-p}$ in $\mathfrak{g}_{\mathbf{C}}$. For $z \in S^1$, the action of $\mathrm{Ad}_{G(\mathbf{R})}(h(z))$ on $\mathfrak{g}_{\mathbf{C}}$ is via $z^{-p}\overline{z}^p = z^{-2p}$ on $V_h^{p,-p}$. Thus, by Lemma 3.5 we have $V_h^{0,0}$ is 2-dimensional and $V_h^{1,-1}$ and $V_h^{-1,1}$ are lines; all other $V_h^{p,-p}$ vanish. In other words, the Hodge structures are all of type $\{(-1,1),(0,0),(1,-1)\}$.

We now assume (G, X) *satisfies Axiom* 0. In particular, for any algebraic representation $\rho : G \to U$, there is a decomposition $\oplus U_n$ of U over $\mathbf{R}$ so that $(U_n)_{\mathbf{C}}$ is the weight-n part of the Hodge decomposition on $U_{\mathbf{C}}$ arising from $\rho_{\mathbf{R}} \circ h$ for *any* $h \in X$. Consider how the Hodge filtration $\{F^p_{n,\rho,h}\}_{p \in \mathbf{Z}}$ on $(U_n)_{\mathbf{C}}$ varies with h. By Theorem 7.1 below, under some additional axioms (Axioms I and II) to be specified later, there is a unique holomorphic structure on X such that for *every* $\rho : G \to \mathrm{GL}(U)$ and $n \in \mathbf{Z}$, inside the trivial holomorphic vector bundle $\mathscr{H}_{n,\rho} = X \times (U_n)_{\mathbf{C}}$ over X the family of subspaces $\{F^p_{n,\rho,h} \subseteq U_n\}_{h \in X}$ for each $p \in \mathbf{Z}$ forms a holomorphic subbundle $\mathscr{F}^p_{n,\rho} \subseteq \mathscr{H}_{n,\rho}$. Since we are only aiming to motivate Deligne's axioms, for now suppose that there is such a holomorphic structure on X. Deligne's Axiom I will concern the extent to which the these subbundles are "varying" within the trivial vector bundle $\mathscr{H}_{n,\rho}$. This is detected by using these subbundles to define a holomorphic map from each connected component of X to a suitable flag variety attached to U_n, and studying the derivative of this map (as a measure of first-order variation of the subbundles). In more direct geometric terms, one has the following result:

Proposition 5.3. *Assume that X satisfies Axiom 0 and admits a holomorphic structure such that for every $\rho : G \to \mathrm{GL}(U)$ and $n \in \mathbf{Z}$, the decreasing Hodge filtrations $\{F^p_{n,\rho,h}\}$ on U_n as h varies in X form the fibers of a decreasing chain of holomorphic subbundles of the trivial vector bundle $X \times (U_n)_{\mathbf{C}} \to X$.*

Identifying holomorphic vector bundles with locally free sheaves of finite rank, consider the derivative map

$$\nabla_\rho = 1 \otimes \mathrm{d} : \mathscr{H}_{n,\rho} := (U_n)_{\mathbf{C}} \otimes_{\mathbf{C}} \mathscr{O}_X \to (U_n)_{\mathbf{C}} \otimes_{\mathbf{C}} \Omega^1_X = \mathscr{H}_{n,\rho} \otimes_{\mathscr{O}_X} \Omega^1_X.$$

Then $\nabla_\rho(\mathscr{F}^p_{n,\rho}) \subseteq \mathscr{F}^{p-1}_{n,\rho} \otimes_{\mathscr{O}_X} \Omega^1_X$ ("Griffiths transversality") for all $p \in \mathbf{Z}$ and all (ρ, U) if and only if for every $h \in X$ the weight-0 Hodge structure $\{V_h^{p,-p}\}_{p \in \mathbf{Z}}$ on $\mathfrak{g}$ associated to $\mathrm{Ad}_{G(\mathbf{R})} \circ h$ is of type

$\{(-1,1),(0,0),(1,-1)\}$ *(i.e.,* $V_h^{p,-p}=0$ *when* $p\neq 0,\pm 1$*). Moreover, it is also equivalent for the transversality condition to hold for a single faithful* ρ*.*

Proposition 5.3 will never be used elsewhere in these notes; its only purpose is to motivate the interest in Deligne's Axiom I below. Before we prove this proposition, let's briefly explain the motivation for considering the Griffiths transversality condition. Consider the ur-Example of a "variation of Hodge structures": the Hodge structures arising in the cohomology of the fibers of a morphism $f: X\to S$ that is the analytification of a smooth proper morphism between smooth algebraic varieties. More specifically, for any such f one defines "relative de Rham cohomology" sheaves $\mathscr{H}^n_{\rm dR}(X/S)=\mathrm{R}^n f_*(f^{-1}\mathscr{O}_S)\simeq \mathrm{R}^n f_*(\mathbf{C})\otimes_{\mathbf{C}}\mathscr{O}_S$, and homological methods provide a relative Hodge to de Rham spectral sequence that defines a natural decreasing filtration on $\mathscr{H}^n_{\rm dR}(X/S)$ by $\mathscr{O}_S$-submodules $\mathscr{F}^p$. A semicontinuity argument shows that these subsheaves are actually locally free with formation commuting with any base change, inducing on s-fibers exactly the traditional Hodge filtration on $\mathrm{H}^n(X_s,\mathbf{C})$ determined by the complex structure on X_s.

Ehresmann's theorem provides a sense in which the $\mathrm{H}^n(X_s,\mathbf{C})$ for fixed n are "locally constant" in s, but the stages of the Hodge filtration usually"move" within this fixed space. This is seen quite explicitly for $n=1$ in the case of the Weierstrass family of elliptic curves over $\mathbf{C}-\mathbf{R}$, for which the line $\mathrm{H}^0(\mathscr{E}_\tau,\Omega^1)=\mathbf{C}\mathrm{d}z$ moves within $\mathrm{H}^1(\mathscr{E},\mathbf{C})=\mathbf{C}\Lambda_\tau^*=\mathbf{C}^2$ via the expression $\mathrm{d}z=\tau e_1^*+e_2^*$ with varying τ. Thus, it is natural to consider the movement of the filtration when varying s as a measure of the "variation of complex structure" in the fibers X_s. This amounts to studying the geometry of the holomorphic map from S to a flag variety as defined by the Hodge filtration $\{\mathscr{F}^p\}$. In this setting, it was a fundamental discovery of Griffiths that these filtrations satisfy the traversality condition relative to

$$\nabla = 1\otimes \mathrm{d}_S : \mathscr{H}^n_{\rm dR}(X/S)=\mathrm{R}^n f_*(\mathbf{C})\otimes_{\mathbf{C}}\mathscr{O}_S\to \mathrm{R}^n f_*(\mathbf{C})\otimes_{\mathbf{C}}\Omega^1_S-\mathscr{H}^n_{\rm dR}(X/S)\otimes_{\mathscr{O}_S}\Omega^1_S.$$

Now we prove Proposition 5.3.

Proof. We can treat the U_n's separately, so we may and do assume that U is a pure Hodge structure of weight n. Any ρ sits inside of a faithful ρ' (e.g., the direct sum of ρ with a faithful representation of G), so by functoriality considerations it suffices to consider the case of faithful ρ.

Fix a choice of faithful ρ, and write $F_h^\bullet$ rather than $F_{h,\rho}^\bullet$. By hypothesis the F_h^p's for varying h and fixed p are the fibers of a holomorphic vector bundle over X, so the dimension of F_h^p depends only on p and the connected component of h in X. (In the proof of Theorem 7.1 we will directly prove this fact from the axioms of a Shimura datum.) For each connected component X_j of X, let $d_j(p)=\dim F_h^p$ for all $h\in X_j$. Let $V=U_{\mathbf{C}}$ and for each j define $\mathrm{Flag}_{d_j}(V)$ to be the flag variety that classifies decreasing flags $F^\bullet$ in V of type d_j: $\dim F^p=d_j(p)$ for all $p\in\mathbf{Z}$. The natural action of $\mathrm{Aut}(V)$ on each $\mathrm{Flag}_{d_j}(V)$ is transitive, and for any specific flag $F^\bullet$ on V of type d_j the stabilizer $\mathrm{Aut}(V)_{F^\bullet}$ at $F^\bullet$ is the group of automorphisms of V that carry each F^p into (and hence isomorphically onto) itself.

The resulting complex-analytic orbit isomorphism $\mathrm{Aut}(V)/\mathrm{Aut}(V)_{F^\bullet}\simeq \mathrm{Flag}_{d_j}(V)$ carrying 1 to $F^\bullet$ (Proposition 14(iii), §1.7, Chapter III of Bourbaki "Lie groups and Lie algebras") induces a tangent space isomorphism

$$\mathrm{End}(V)/\mathrm{End}_{F^\bullet}(V)\simeq \mathrm{T}_{F^\bullet}(\mathrm{Flag}_{d_j}(V))$$

where $\mathrm{End}_{F^\bullet}(V)$ is the space of endomorphisms T of V such that $T(F^p)\subseteq F^p$ for all $p\in\mathbf{Z}$. (This is most easily seen by working algebro-geometrically and considering points valued in the dual numbers over $\mathbf{C}$.)

Now fix a connected component X_j of X and consider the natural holomorphic map $\phi_j: X_j\to \mathrm{Flag}_{d_j}(V)$ defined by $h\mapsto F_h^\bullet$. We claim that this map is injective. That is, we can reconstruct h from the Hodge filtration $\{F_h^p\}$ on $U_{\mathbf{C}}$. Indeed, $V^{p,q}=F_h^p\cap\overline{F}_h^q$ due to purity, so from the Hodge filtration we can reconstruct the Hodge decomposition. But under the correspondence in Proposition 4.9 (which underlies how the Hodge structure on U associated to h is *defined*), the action of $\rho(h(z))_{\mathbf{C}}$ on $U_{\mathbf{C}}$ is via $z^{-p}\overline{z}^{-q}$ on $V^{p,q}$. Hence, from the $V^{p,q}$'s we can determine each $\rho(h(z))_{\mathbf{C}}$ for $z\in\mathbf{C}^\times$ and hence can determine h (as we have arranged for ρ to be *faithful*). With the injectivity of ϕ_j established, we now invoke a basic general fact from the theory of holomorphic functions in several variables: an injective holomorphic map between complex manifolds necessarily has injective derivative at all points in a dense open set. (In the 1-dimensional case this is the familiar classical fact that if an analytic function in one complex variable has a zero of order n at a point then it is n-to-1 on a punctured disc around the point, so injectivity of the function forces injectivity of its derivative maps.) Thus, $\mathrm{d}\phi_j(h)$ is injective for many $h \in X_j$. By equivariance of ϕ_j with respect to the open subgroup of $G(\mathbf{R})$ that preserves X_j, it follows that $\mathrm{d}\phi_j(h)$ is injective for all $h \in X_j$.

Choose $h \in X$ and identify X with $G(\mathbf{R})/Z_h$ This identifies $\mathrm{T}_h(X)$ with $\mathfrak{g}/\mathfrak{g}^{0,0}$ (where $\mathfrak{g} = \mathrm{Lie}(G(\mathbf{R}))$ and $\mathfrak{g}^{0,0}$ is the $\mathbf{R}$-descent of the space $\mathfrak{g}_{\mathbf{C}}^{0,0}$ defined using the Hodge structure on $\mathfrak{g}$ defined by $\mathrm{Ad}_{G(\mathbf{R})} \circ h : \mathbf{C}^\times \to G(\mathbf{R}) \to \mathrm{Aut}(\mathfrak{g})$). Thus, $\mathrm{d}\phi_j(h)$ is identified with an $\mathbf{R}$-linear map

$$\mathfrak{g}/\mathfrak{g}^{0,0} \to \mathrm{End}(V)/\mathrm{End}_{F_h^\bullet}(V).$$

The map ϕ_j satisfies

$$\phi_j(h(z)gh(z)^{-1}) = \rho(h(z))_{\mathbf{C}}(\rho(g)_{\mathbf{C}}(\rho(h(z))_{\mathbf{C}}^{-1}(F_h^\bullet))) = \rho(h(z))_{\mathbf{C}}(\rho(g)_{\mathbf{C}}(F_h^\bullet)) = \rho(h(z))_{\mathbf{C}}(F_{g.h}^\bullet),$$

where the final equality rests on the fact that $\rho(g)_{\mathbf{C}}(V_h^{p,q}) = V_{g.h}^{p,q}$. This latter identity holds because for any $v \in V_h^{p,q}$ and $z \in \mathbf{C}^\times$ we have

$$\rho((g.h)(z))(\rho(g)_{\mathbf{C}}(v)) = \rho(g)_{\mathbf{C}}(\rho(h(z))_{\mathbf{C}}(v)) = \rho(g)_{\mathbf{C}}(z^{-p}\overline{z}^{-q} \cdot v) = z^{-p}\overline{z}^{-q} \cdot \rho(g)_{\mathbf{C}}(v)$$

(since $(g.h)(z) := g \circ h(z)g^{-1}$ in G). In other words, ϕ_j intertwines the $\mathbf{C}^\times$-action on G via h with the $\mathbf{C}^\times$-action on $\mathrm{Flag}_{d_j}(V)$ via $\rho_{\mathbf{C}} \circ h$ (where $\rho_{\mathbf{C}}$ denotes ρ restricted to $\mathbf{C}$-points). Passing to the derivative map between tangent spaces at $1 \in G$ and $F_h^\bullet \in \mathrm{Flag}_{d_j}(V)$, it follows that the tangent map $\mathrm{d}\phi_j(h) : \mathfrak{g}/\mathfrak{g}^{0,0} \to \mathrm{End}(V)/\mathrm{End}_{F_h^\bullet}(V)$ intertwines the adjoint action of $\mathbf{C}^\times$ via h on the source with the action through conjugation via $\rho_{\mathbf{C}} \circ h$ on the target (because under the orbit map $\mathrm{Aut}(V) \twoheadrightarrow \mathrm{Flag}_{d_j}(V)$ defined by $T \mapsto T(F_h^\bullet)$ intertwines the left action by $(\rho \circ h)(z)_{\mathbf{C}}$ on the target with conjugation by $(\rho \circ h)(z)_{\mathbf{C}}$ on $\mathrm{Aut}(V)$, as $F_h^\bullet$ is fixed by $(\rho \circ h)(z)_{\mathbf{C}}^{-1}$).

In other words, $\mathrm{d}\phi_j(h)$ is compatible with the Hodge structure on $\mathfrak{g}$ induced by h (via the adjoint action of $G(\mathbf{R})$ on $\mathfrak{g}$) and the Hodge structure on $\mathrm{End}(U)$ induced by the Hodge structure on U via $\rho \circ h$ (see Example 4.8). By Example 4.8, the subspace $\mathrm{End}_{F_h^\bullet}(V)$ of $\mathrm{End}(V) = \mathrm{End}(U)_{\mathbf{C}}$ is precisely $F^0(\mathrm{End}(V))$ relative to the Hodge structure on $\mathrm{End}(U)$ induced by $\rho_{\mathbf{R}} \circ h$. On fibers at $h \in X_j$, the transversality condition $\nabla_\rho(\mathscr{F}_\rho^p) \subseteq \mathscr{F}_\rho^{p-1} \otimes_{\mathscr{O}_X} \Omega_X^1$ says exactly that in any first-order direction in X at h the subbundle filtration $\{\mathscr{F}_\rho^p\}$ deforms each F_h^p at most within the subspace F_h^{p-1}. That is, the image of $\mathrm{d}\phi_j(h)$ within $\mathrm{End}(V)/F^0(\mathrm{End}(V))$ lies in the coset classes of those $T \in \mathrm{End}(V)$ such that $T(F_h^p) \subseteq F_h^{p-1}$ for all $p \in \mathbf{Z}$, which is exactly $F^{-1}(\mathrm{End}(V))$ (see Example 4.8).

We have proved the equivalence between the transversality condition at h and the condition that the map $\mathrm{d}\phi_j(h)$ lands inside $F^{-1}(\mathrm{End}(V))/F^0(\mathrm{End}(V))$. But the $\mathbf{C}$-linear extension

$$\mathfrak{g}_{\mathbf{C}}/\mathfrak{g}_{\mathbf{C}}^{0,0} = (\mathfrak{g}/\mathfrak{g}^{0,0})_{\mathbf{C}} \to \mathrm{End}(V)/\mathrm{End}_{F_h^\bullet}(V)$$

is *injective* (since it is $\mathrm{d}\rho(1)$, and ρ is faithful), so it lands in $F^{-1}(\mathrm{End}(V))/F^0(\mathrm{End}(V))$ if and only if the $\mathbf{C}^\times$-action on $\mathfrak{g}/\mathfrak{g}^{0,0}$ via the adjoint action of h makes the complexification be exactly $F^{-1}(\mathfrak{g}_{\mathbf{C}})/\mathfrak{g}_{\mathbf{C}}^{0,0}$. Equivalently, it is necessary and sufficient that $\mathfrak{g}_{\mathbf{C}} = F^{-1}(\mathfrak{g}_{\mathbf{C}})$, which is to say that the Hodge structure on $\mathfrak{g}$ defined by h makes $\mathfrak{g}_{\mathbf{C}}^{p,q}$ vanish whenever $p < -1$.

By Axiom 0 the Hodge structure on $\mathfrak{g}$ defined by h has weight 0, so the only possibly nonzero $\mathfrak{g}_{\mathbf{C}}^{p,q}$'s are for $q = -p$. Since complex conjugation on $\mathfrak{g}_{\mathbf{C}}$ swaps $\mathfrak{g}_{\mathbf{C}}^{q,p}$ and $\mathfrak{g}_{\mathbf{C}}^{p,q}$, we conclude that the transversality condition on fibers at $h \in X$ is equivalent to the vanishing of $\mathfrak{g}_{\mathbf{C}}^{p,-p}$ whenever $p \neq 0, \pm 1$, which is to say that the Hodge structure on $\mathfrak{g}$ defined by h has type $\{(1,-1),(0,0),(-1,1)\}$. ■

For our GL_2 example, where the Hodge structure comes from the homology of elliptic curves, the Hodge filtration has only two steps. For two-step filtrations the Griffiths transversality condition always holds (since over a connected component of the base, if $\mathscr{F}^p$ is nonzero then $\mathscr{F}^{p-1}$ is the entire ambient vector bundle). Likewise, in Example 5.2 we gave a direct proof that our GL_2 example satisfies the condition on the Hodge type at the end of Proposition 5.3. In fact, the calculations in that example yield the following result that finally provides a Lie-theoretic characterization – up to inversion on $\mathbf{C}^\times$ – of the conjugacy class of maps X in our running GL_2 example:

Proposition 5.4. *Let $G = \mathrm{GL}_2$ and $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{gl}_2(\mathbf{R})$. Up to composing with inversion on $\mathbf{C}^\times$, a map of Lie groups $h : \mathbf{C}^\times \to G(\mathbf{R})$ arises from an $\mathbf{R}$-algebra embedding $\mathbf{C} \to \mathrm{Mat}_2(\mathbf{R})$ if and only if it is algebraic and the resulting Hodge structure on $\mathfrak{g}$ satisfies the following properties: it is pure of weight 0, it has Hodge type $\{(-1,1),(0,0),(1,-1)\}$ (i.e., $V_h^{p,-p} = 0$ when $p \neq 0, \pm 1$), and the weight homomorphism $w_h : \mathrm{GL}_1 \to Z_G$ over $\mathbf{R}$ is an isomorphism (i.e., $w_h : x \mapsto x^{\pm 1}$ via the canonical identification of Z_G with GL_1). The collection of such h with a common weight homomorphism is a single $G(\mathbf{R})$-conjugacy class.*

Note that from a pure group-theoretic viewpoint (without bringing in the $\mathbf{R}$-algebra $\mathrm{Mat}_2(\mathbf{R})$ whose unit group is $G(\mathbf{R})$), the two isomorphisms of Z_G with GL_1 cannot be distinguished and so the effect of inversion on $\mathbf{C}^\times$ cannot be avoided. In this sense, the preceding result is "best possible" with only group-theoretic notions. The conjugacy class consisting of those h whose weight homomorphism $\mathrm{GL}_1 \to Z_G$ is the canonical isomorphism using the matrix realization of G inside of GL_2 is precisely the class X of those h arising from $\mathbf{R}$-algebra embeddings $\mathbf{C} \to \mathrm{Mat}_2(\mathbf{R})$. The conjugacy class whose weight homomorphism is inversion on GL_1 corresponds to the dual Hodge structures on $\mathbf{R}^2$.

Axiom I. *For every $h \in X$, the Hodge structure on $\mathfrak{g}$ via $\mathrm{Ad}_{G(\mathbf{R})} \circ h$ is of type $\{(-1,1),(0,0),(1,-1)\}$.*

Remark 5.5. Axiom I implies Axiom 0, since $V_h^{p,q}$ lies in the part of weight $p+q$.

The next, and perhaps most crucial, axiom is a generalization of the classical observation in the GL_2-case that for every $h \in X$ the centralizer of h in $\mathrm{GL}_2(\mathbf{R})$ meets $\mathrm{SL}_2(\mathbf{R})$ in a maximal compact subgroup. Assuming Axioms I holds, it is convenient to focus on the centralizer of the element $h(i)$ (or $h(-i) = h(-1)h(i)$, which comes to the same thing since $h(-1)$ is central in $G(\mathbf{R})$ by Axiom 0). Up to connectedness, this is the same as the centralizer of the map h in $G(\mathbf{R})$, due to Axiom I:

Proposition 5.6. *Assume (G, X) satisfies Axiom I. For any $h \in X$ and either choice of $i = \sqrt{-1} \in \mathbf{C}$, the centralizer of $h(i)$ in $G(\mathbf{R})$ has the same identity component as the $G(\mathbf{R})$-centralizer of the map h (or equivalently, the centralizer of the image of $h(S^1)$ in $G(\mathbf{R})$).*

Proof. As a warm-up, here is a direct proof in the GL_2-case. Any $T \in G(\mathbf{R}) \subset \mathrm{Mat}_2(\mathbf{R})$ that centralizes $h(i)$ must centralize $\mathbf{R} \oplus \mathbf{R}h(i)$, which in turn is the chosen embedded $\mathbf{C}$ in $\mathrm{Mat}_2(\mathbf{R})$. Thus, T corresponds to a $\mathbf{C}$-linear endomorphism of $\mathbf{R}^2$ equipped with this 1-dimensional $\mathbf{C}$-vector space structure, so $T \in h(\mathbf{C}^\times)$.

In general, look at the adjoint action of $h(i)$ on $V = \mathfrak{g}_{\mathbf{C}} = V_h^{-1,1} \oplus V_h^{0,0} \oplus V_h^{1,-1}$. This is trivial on $V_h^{0,0}$ and is negation on $V_h^{-1,1}$ and $V_h^{1,-1}$. But the centralizer Z of $h(i)$ in $G(\mathbf{R})$ is a closed subgroup such that $\mathrm{Lie}(Z)$ is the locus of trivial adjoint action for $h(i)$, which is to say $\mathrm{Lie}(Z) = \mathfrak{g}^{0,0}$. The $h(S^1)$-action on $G(\mathbf{R})$ by conjugation induces an adjoint action on $\mathfrak{g}$ that preserves $\mathfrak{g}^{0,0} = \mathrm{Lie}(Z)$ with trivial action and hence preserves Z^0 with trivial action. That is, $h(S^1)$ centralizes Z^0. Equivalently, Z^0 is contained in $Z_{G(\mathbf{R})}(h(S^1))$. But $\mathrm{Lie}(Z_{G(\mathbf{R})}(h(S^1)))$ is the subspace of fixed vectors in $\mathfrak{g}$ for the adjoint action of $h(S^1)$, and this subspace is exactly $\mathfrak{g}^{0,0} = \mathrm{Lie}(Z)$, so $Z_{G(\mathbf{R})}(h(S^1))^0 = Z^0$. ■

Remark 5.7. The action on X by $h(i)$ (which preserves the point $h \in X$) induces negation on $\mathrm{T}_h(X)$. To prove this, first use the base point h to identify X with $G(\mathbf{R})/Z_h$ carrying the left action by any $h(z)$ over to the adjoint action by $h(z)$ (for any $z \in \mathbf{C}^\times$). Hence, the $h(z)$-action on $\mathrm{T}_h(X)$ is identified with the adjoint action of $h(z)$ on $\mathfrak{g}/\mathrm{Lie}(Z_h)$ for $\mathfrak{g} := \mathrm{Lie}(G(\mathbf{R}))$. By Axiom I, the Hodge structure determined by h is of type $\{(1,-1),(0,0),(-1,1)\}$, so the adjoint action by $h(i)$ on $V = \mathfrak{g}_{\mathbf{C}}$ is trivial on $V^{0,0}$ and negation on $V^{1,-1}$ and $V^{-1,1}$ (since for $\epsilon = \pm 1$ and $z \in S^1$ we have $z^{\epsilon}\overline{z}^{-\epsilon} = z^{-2\epsilon}$ for $z \in S^1$). By Proposition 5.6 the subspace $\mathrm{Lie}(Z_h)$ in $\mathfrak{g}$ is the locus where the adjoint action of $h(i)$ is trivial, so its complexification is $V^{0,0}$. Hence, on $\mathfrak{g}/\mathrm{Lie}(Z_h)$ the adjoint action of $h(i)$ is indeed negation.

The action of $h(i)$ also naturally arises in the following Hodge-structure generalization of the Weil pairings for elliptic curves and abelian varieties:

Definition 5.8. Let U be a finite-dimensional $\mathbf{R}$-vector space equipped with a Hodge structure $\{V^{p,q}\}$ that is pure of weight n. A *polarization* of this Hodge structure is an $\mathbf{R}$-bilinear form

$$\psi : U \times U \to \mathbf{R}(-n)$$

such that $\psi(z.u, z.u') = \psi(u, u')$ for $z \in S^1 \subset \mathbf{C}^\times$ and the pairing

$$\langle u, u'\rangle \mapsto (2\pi i)^n \psi(u, h(i)u') \in \mathbf{R}$$

a symmetric positive-definite $\mathbf{R}$-bilinear form.

Remark 5.9. The S^1-equivariance hypothesis says exactly that the induced $\mathbf{R}$-linear map $U \otimes_{\mathbf{R}} U \to \mathbf{R}(-n)$ between pure Hodge structures of weight $2n$ is $\mathbf{C}^\times$-equivariant, or equivalently a morphism of Hodge structures. Also, note that $\langle \cdot, \cdot \rangle$ is independent of the choice of $i = \sqrt{-1} \in \mathbf{C}$, since $h(-1) = (-1)^{-n} = (-1)^n$ (due to the purity hypothesis on the Hodge structure), and ψ is $(-1)^n$-symmetric since symmetry of $\langle \cdot, \cdot \rangle$ implies that $\psi(u, h(i)u') = \psi(u', h(i)u) = \psi(h(i)u', h(-1)u) = (-1)^n \psi(h(i)u', u)$.

Example 5.10. Let $E = \mathbf{C}/\Lambda_\tau$ and consider the Hodge structure on $U = \mathrm{H}_1(E, \mathbf{R})$ associated to its usual complex structure (via identification with $\mathrm{T}_0(E)$). This is weight -1, and there is a natural $\mathbf{R}$-bilinear form

$$\mathrm{H}_1(E, \mathbf{Z}) \times \mathrm{H}_1(E, \mathbf{Z}) \to \mathbf{Z}(1)$$

defined by $(c, c') \mapsto 2\pi i I_i(c, c')$ where $i = \sqrt{-1} \in \mathbf{C}$ and $I_i(c, c')$ denotes the i-oriented intersection pairing (which depends up to a sign on the choice of $i = \sqrt{-1} \in \mathbf{C}$, so multiplying against $2\pi i$ makes the output independent of this choice). Let's check that the $\mathbf{R}$-scalar extension is a polarization. The S^1-invariance condition amounts to the fact that in the tangent planes of E (equipped with their $\mathbf{C}$-line structure), rotation by an angle has no effect on the intersection number for two oriented curves meeting transversally at a point. The intersection pairing is visibly skew-symmetric, so the pairing $\langle u, u'\rangle = I_i(u, iu')$ is symmetric.

Let's compute the quadratic form associated to this symmetric bilinear form in order to verify that it is positive-definite. We pick the unique $i = \sqrt{-1}$ such that τ is in the i-component of $\mathbf{C} - \mathbf{R}$, so $\tau = a + bi$ with $b > 0$. Then $i = (\tau - a)/b$, so since $I_i(1, 1) = I_i(\tau, \tau) = 0$ and $I_i(1, \tau) = 1$ (as $b > 0$!) and $I_i(\tau, 1) = -1$, we find

$$\langle 1, 1\rangle = I_i(1, i) = 1/b \quad \langle 1, \tau\rangle = \langle \tau, 1\rangle = I_i(\tau, i) = a/b,$$

and

$$\langle \tau, \tau\rangle = I_i(\tau, i\tau) = I_i(\tau, -b + ai) = b + aI_i(\tau, i) = b + a^2/b = (a^2 + b^2)/b.$$

Thus, the quadratic form is

$$\frac{1}{b}(x^2 + 2axy + (a^2 + b^2)y^2) = \frac{(x + ay)^2 + b^2y^2}{b},$$

which is positive-definite on $\mathbf{R}^2$ because $b > 0$.

Example 5.11. Projective embeddings gives rise to polarizations of the Hodge structure on the $\mathbf{C}$-valued cohomology of a smooth projective complex manifold. This expresses the so-called Hodge–Riemann bilinear relations, which are a vast generalization of the Riemann relations in the Abel–Jacobi theorem for compact Riemann surfaces.

For any $h \in X$, the conjugation action by $h(i)$ on G is independent of the choice of $i = \sqrt{-1} \in \mathbf{C}$ and is an involution since $h(-1)$ is central in G (by the weight-0 hypothesis). Denote this involution as ι_h. Note that ι_h is nontrivial whenever X is not a single point (i.e., the elements $h \in X$ are not centralized by $G(\mathbf{R})$, which is equivalent to $\mathrm{Lie}(\iota_h) \neq 1$ on $\mathfrak{g}$ since G is a connected $\mathbf{R}$-group and $G(\mathbf{R})$ is Zariski-dense in G). Reductive groups naturally make their appearance in the following result.

Proposition 5.12. *Assume (G, X) satisfies Axiom I. Consider all representations $\rho : G \to \mathrm{GL}(U)$ and the resulting Hodge structures $\{V_{n,h}^{p,q}\}$ on U_n for $h \in X$. In order that every such Hodge structure admits a polarization, it is necessary and sufficient that the following two conditions hold.*

1. *Let G_1 be the minimal connected closed $\mathbf{R}$-subgroup in G such that all maps $h : \mathbf{C}^\times \to G(\mathbf{R})$ factor through $G_1(\mathbf{R})$. Then G_1 is reductive.*
2. *Let G_1' be the connected semisimple "derived group" of G_1 (e.g., $G_1' = \mathrm{SL}_n$ when $G_1 = \mathrm{GL}_n$). Then the fixed-point locus of ι_h in $G(\mathbf{R})$ meets $G_1'(\mathbf{R})^0$ in a maximal compact subgroup for some (equivalently, any) $h \in X$.*

Proof. For a proof, see Proposition 1.1.14(iii) in Deligne's article on Shimura varieties in volume 2 of the Corvallis proceedings. We will only use this result for motivational purposes in connection with Axiom II below. Note that in our running example with GL_2, we have $G_1 = G = \mathrm{GL}_2$, so $G_1' = \mathrm{SL}_2$. Thus, both conditions (1) and (2) above are satisfied in this case. ■

In general, since X is a $G(\mathbf{R})$-conjugacy class it follows that G_1 is normalized by $G(\mathbf{R})$ and hence is a *normal* $\mathbf{R}$-subgroup of G since $G(\mathbf{R})$ is Zariski-dense in G. The elements of X are all valued in $G_1(\mathbf{R})$, and an argument using the structure theory of reductive groups shows that the elements of X are $G_1(\mathbf{R})$-conjugate to each other (and hence constitute a $G_1(\mathbf{R})$-conjugacy class). The general fact being invoked here is that if G is a connected linear algebraic group over a field k of characteristic 0 and if N is a normal connected reductive k-subgroup of G containing a pair of $G(k)$-conjugate connected reductive k-subgroups H and H' (such as $N = G_1$ in the case of interest over $\mathbf{R}$, and H and H' the respective $\mathbf{R}$-tori corresponding to $h(\mathbf{C}^\times)$ and $h'(\mathbf{C}^\times)$ for a pair of elements $h, h' \in X$), then H and H' are $N(k)$-conjugate. Thus, by replacing G with G_1 we do not affect X and so are led to:

Axiom II. *The connected linear algebraic* $\mathbf{R}$*-group* G *is reductive, and for the semisimple derived group* G' *and some (equivalently, any)* $h \in X$ *the action by* ι_h *on* $G'(\mathbf{R})^0$ *has fixed-point locus equal to a maximal compact subgroup of* $G'(\mathbf{R})^0$.

By Proposition 5.12, the fixed-point condition in Axiom II is equivalent to the property that the pure component of every Hodge structure arising from an $h \in X$ and an algebraic representation of G admits a polarization. Using Proposition 5.6 and facts in the theory of reductive groups, in Axiom II it is equivalent to replace consideration of the fixed-point locus of ι_h with consideration of the centralizer of $h(S^1)$ (as in our initial viewpoint on the GL_2-case).

The following general concept applies to the pair (GL_2, X) arising from $\mathbf{R}$-algebra embeddings $\mathbf{C} \to \mathrm{Mat}_2(\mathbf{R})$, but with GL_2 viewed as a $\mathbf{Q}$-group rather than as an $\mathbf{R}$-group.

Definition 5.13. Consider a pair (G, X) where G is a connected reductive $\mathbf{Q}$-group and X is a $G(\mathbf{R})$-conjugacy class of algebraic homomorphisms $h : \mathbf{C}^\times \to G$. Let Z_G denote the center of G (in the sense of algebraic groups over $\mathbf{Q}$). The pair (G, X) is a *Shimura datum* when:

- Axioms I and II hold for the pair $(G_{\mathbf{R}}, X)$,
- there is no proper closed normal $\mathbf{Q}$-subgroup of G/Z_G whose $\mathbf{R}$-points contain the image of every $h \in X$.

The point of Axioms I and II has been explained above. The new condition concerning $\mathbf{Q}$-subgroups is a purely technical "minimality" requirement on G with respect to X, intended to avoid some silly situations. (It precisely avoids the possibility of G containing a nontrivial $\mathbf{Q}$-simple isogeny factor H such that $H(\mathbf{R})$ centralizes all $h \in X$. In such cases we could generally replace G with the $\mathbf{Q}$-subgroup obtained by removing H from the isogeny decomposition of the derived group of G without any significant effect on X, as is made precise by Corollary 6.7.) This condition will play no role in what follows.

The $\mathbf{Q}$-structure on G is a new ingredient: it has not been relevant in what has gone before (where only an $\mathbf{R}$-group was needed), but in the next section we explain the importance of a $\mathbf{Q}$-structure on G, and more specifically a special class of subgroups of $G(\mathbf{Q})$ whose action on X is essential to the arithmetic interest of the concept of a Shimura datum.

Remark 5.14. In the theory of Hilbert modular forms, one works with the algebraic group GL_2 over a totally real field F. This theory is encoded in terms of a Shimura datum, generalizing the case of elliptic modular forms (i.e., $F = \mathbf{Q}$) by using the $\mathbf{Q}$-group $G = \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_2)$ and the $G(\mathbf{R})$-conjugacy class of maps $h : \mathbf{C}^\times \to G(\mathbf{R}) = \prod_{v|\infty} \mathrm{GL}_2(\mathbf{R})$ arising from an $\mathbf{R}$-algebra embedding $\mathbf{C} \to \prod_{v|\infty} \mathrm{Mat}_2(\mathbf{R})$. Note that since

$$\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_2)_{\overline{\mathbf{Q}}} \simeq \prod_{\sigma : F \to \overline{\mathbf{Q}}} \mathrm{GL}_2$$

as $\overline{\mathbf{Q}}$-groups, a key feature of the theory of Shimura data that $G_{\overline{\mathbf{Q}}}$ may have several simple isogeny factors even when G is $\mathbf{Q}$-simple.

6. Arithmetic subgroups and quotients

Fix a Shimura datum (G, X), so X is made into a real-analytic manifold via the identification with $G(\mathbf{R})/Z_h$ for any $h \in X$, where Z_h denotes the $G(\mathbf{R})$-centralizer of h. Even better, X is canonically equipped with a structure of complex manifold via Theorem 7.1, but we will not use that fact in this section. Let G' denote the derived group of G (e.g., $G' = \mathrm{SL}_n$ when $G = \mathrm{GL}_n$), so the theory of reductive groups implies that the natural map of $\mathbf{Q}$-groups $G' \to G/Z_G$ is an isogeny (e.g., $\mathrm{SL}_n \to \mathrm{PGL}_n$ for $G = \mathrm{GL}_n$). The main interest in Shimura data (G, X) is for the study of quotients $\Gamma\backslash X$ of X by the left action of a distinguished class of subgroups Γ of $G'(\mathbf{Q})$ that are discrete in $G'(\mathbf{R})$. This class is defined as follows over any global field:

Definition 6.1. Let H be a smooth affine group over a global field k, and let S be a finite non-empty set of places of k that contains the archimedean places. A subgroup Γ in $H(k)$ is *S-arithmetic* if it is commensurable inside $H(k)$ with $\mathscr{H}(\mathscr{O}_{k,S})$ for some flat affine finite type $\mathscr{O}_{k,S}$-group $\mathscr{H}$ with generic fiber H. If k is a number field and S is the set of archimedean places (so $\mathscr{O}_{k,S} = \mathscr{O}_k$) then we say *arithmetic* to mean S-arithmetic.

Example 6.2. If $k = \mathbf{Q}$ and $S = \{\infty, p|N\}$ for some integer $N > 0$ then $\mathrm{SL}_n(\mathbf{Z}[1/N])$ is an S-arithmetic subgroup of $\mathrm{SL}_n(\mathbf{Q})$.

In the definition of S-arithmeticity it turns out that the choice of $\mathscr{H}$ does not matter (i.e., the commensurability condition holds for all choices if it holds for one choice), and such S-integral models can be constructed rather concretely: we pick a closed k-subgroup inclusion of H into some GL_n and then take $\mathscr{H}$ to be the Zariski-closure of H in the $\mathscr{O}_{k,S}$-group GL_n (in which case $\mathscr{H}(\mathscr{O}_{k,S})$ is simply $\mathrm{GL}_n(\mathscr{O}_{k,S}) \cap H(k)$). Using that $\mathscr{O}_{k,S}$ is Dedekind, it can be proved that all choices of $\mathscr{H}$ arise in this way, so the definition of S-arithmeticity is equivalent to the condition of commensurability with $H(k) \cap \mathrm{GL}_n(\mathscr{O}_{k,S})$ for some faithful representation $H \to \mathrm{GL}_n$ over k (which is the version of the definition that one usually finds in the literature).

Remark 6.3. When k is a number field, the arithmetic subgroups of $H(k)$ are always discrete in the Lie group $H(\mathbf{R} \otimes_{\mathbf{Q}} k) = \prod_{v|\infty} H(k_v)$. Indeed, discreteness can be checked using a finite-index subgroup, so for a flat affine finite type $\mathscr{O}_k$-group $\mathscr{H}$ with generic fiber H we just have to check that $\mathscr{H}(\mathscr{O}_k)$ is discrete in $H(\mathbf{R} \otimes_{\mathbf{Q}} k) = \mathscr{H}(\mathbf{R} \otimes_{\mathbf{Q}} k)$. Using a closed immersion of $\mathscr{H}$ into an affine space over $\mathscr{O}_k$, this reduces to the analogous discreteness claim for such an affine space in place of $\mathscr{H}$. That in turn is a consequence of the evident discreteness of $\mathscr{O}_k$ in $\mathbf{R} \otimes_{\mathbf{Q}} k = \mathbf{R} \otimes_{\mathbf{Z}} \mathscr{O}_k$.

Example 6.4. Suppose k is a number field and Γ is an S-arithmetic subgroup of $H(k)$ for a smooth affine k-group H. We claim that Γ contains a finite-index normal subgroup Γ' that is torsion-free. It suffices to find such a Γ' without the normality condition (as any finite-index subgroup of a group always contains a finite-index subgroup that is normal in the ambient group). Choose an embedding of H as a closed k-subgroup of some GL_n, so Γ is commensurable with $H(k) \cap \mathrm{GL}_n(\mathscr{O}_{k,S})$. Thus, it suffices to find a torsion-free finite-index subgroup of $\mathrm{GL}_n(\mathscr{O}_{k,S})$. In fact, we claim that the congruence subgroup

$$\{g \in \mathrm{GL}_n(\mathscr{O}_{k,S}) \,|\, g \equiv 1 \bmod N\}$$

is torsion-free for any $N \geq 3$.

To see this, first note that if g has finite order then its eigenvalues are roots of unity, and moreover $g = 1$ if and only if these eigenvalues are 1 (since a unipotent matrix of finite order is trivial in characteristic 0). Thus, it suffices to prove that if $\zeta \in \overline{\mathbf{Q}}^{\times}$ is a root of unity and $\zeta \equiv 1 \bmod N\overline{\mathbf{Z}}$ with $N \geq 3$ then $\zeta = 1$. Since N is divisible by 4 or an odd prime, it suffices to verify the p-adic analogue with $N = p$ for odd p and $N = 4$ for $p = 2$. In all cases $\zeta - 1$ is in the domain where $\log_p(1 + x)$ is convergent with inverse $\exp_p$, yet $\log_p(1 + (\zeta - 1)) = 0$ since ζ is a root of unity, so we are done.

Remark 6.5. An advantage of the "abstract" definition of S-arithmeticity (without reference to matrix realizations) is that it leads to a nice proof of the preservation of S-arithmeticity under *central* isogenies $f : H \to H'$ between connected reductive k-groups (a typical example being $\mathrm{SL}_n \to \mathrm{PGL}_n$). The meaning of this claim is that an S-arithmetic subgroup of $H(k)$ has S-arithmetic image in $H'(k)$, and conversely any S-arithmetic subgroup of $H'(k)$ has S-arithmetic preimage in $H(k)$. The basic issue is that although $H(k) \to H'(k)$ has finite kernel and its image is normal (due to the *central* quotient H' of H naturally acting on H compatibly with the conjugation action of H' on itself), the quotient $H'(k)/H(k)$ is typically infinite (e.g., for $\mathrm{SL}_n \to \mathrm{PGL}_n$ this quotient is $k^\times/(k^\times)^n$). In general, letting $Z = \ker f$, the quotient $H(k)/H'(k)$ is a subgroup of the cohomology group $\mathrm{H}^1(k, Z)$ (Galois cohomology if f is étale, such as when k is a number field, and fppf cohomology in general). This cohomology group is usually infinite. By using the abstract definition of S-arithmeticity, one can reduce the preservation of S-arithmeticity under f to the fact that an analogous S-integral cohomology is finite, and that in turn can be reduced to the finiteness of $\mathscr{O}_{k,S'}^\times/(\mathscr{O}_{k,S'}^\times)^n$ for any finite S' containing S, which follows from the classical S'-unit theorem.

In order that $\Gamma\backslash X$ be a reasonable object, the Γ-action on X should be discontinuous, and in particular the isotropy groups $\Gamma_h = \Gamma \cap Z_h$ should be finite. Beware that when X is equipped with the complex structure to be constructed in Theorem 7.1, if $\dim_{\mathbf{C}} X > 1$ then the quotients $\Gamma\backslash X$ will generally not exist in the sense of complex manifolds if there are nontrivial isotropy groups – they will merely be normal complex-analytic spaces. The failure of smoothness is quite mild, since Example 6.4 implies that the group Γ always contains a *torsion-free* (even normal) subgroup Γ' of finite index and so we can apply:

Proposition 6.6. *The action by Γ' on X is properly discontinuous.*

Proof. The group $G(\mathbf{R})$ has finitely many connected components (a general fact for any linear algebraic $\mathbf{R}$-group, or even any smooth $\mathbf{R}$-scheme of finite type), so the topological space $X = G(\mathbf{R})/Z_h$ (for $h \in X$) has only finitely many connected components. Hence, the Γ'-stabilizer Γ'_0 of a connected component X_0 of X has finite index. Thus, it suffices to show that Γ'_0 acts properly discontinuously on X_0. We fix a choice of X_0 and base point $h_0 \in X_0$, so there is a $G(\mathbf{R})^0$-equivariant isomorphism $G(\mathbf{R})^0/(Z_{h_0} \cap G(\mathbf{R})^0) \simeq X_0$. The key is to show that the natural map

$$G'(\mathbf{R})^0/(Z_{h_0} \cap G'(\mathbf{R})^0) \to G(\mathbf{R})^0/(Z_{h_0} \cap G(\mathbf{R})^0) = X_0 \tag{6.1}$$

is a (real-analytic) isomorphism. Granting this fact, we conclude as follows.

The key is to prove that $Z_{h_0} \cap G'(\mathbf{R})^0$ is a maximal compact subgroup of $G'(\mathbf{R})^0$. To establish this property, first observe that $Z_{h_0} \cap G'(\mathbf{R})^0$ has the same identity component as the locus of fixed points for ι_{h_0} on $G'(\mathbf{R})^0$ (as it suffices to check the analogue with $G'(\mathbf{R})^0$ removed, so Proposition 5.6 applies). Thus, by Axiom II, the identity component of $Z_{h_0} \cap G'(\mathbf{R})^0$ is the same as the identity component of a maximal compact subgroup of $G'(\mathbf{R})^0$. But it is a general fact that the maximal compact subgroups in the identity component of the $\mathbf{R}$-points of any connected semisimple $\mathbf{R}$-group (such as G') are *connected*, so the identity component of $Z_{h_0} \cap G'(\mathbf{R})^0$ is a maximal compact subgroup K of $G'(\mathbf{R})^0$. The closed subgroup Z_{h_0} in $G(\mathbf{R})$ is the $\mathbf{R}$-points of a Zariski-closed $\mathbf{R}$-subgroup of G (Remark 3.3), so $Z_{h_0} \cap G'(\mathbf{R})$ is likewise the $\mathbf{R}$-points of a linear algebraic $\mathbf{R}$-group. This implies that its identity component has *finite* index, so the identity component of $Z_{h_0} \cap G'(\mathbf{R})^0$ has finite index. But this latter identity component is the compact K, so $Z_{h_0} \cap G'(\mathbf{R})^0$ is compact. Maximality of K then forces $Z_{h_0} \cap G'(\mathbf{R})^0 = K$.

To summarize, (6.1) identifies X_0 with $G'(\mathbf{R})^0/K$ for a maximal compact subgroup K of $G'(\mathbf{R})^0$. This identification is $G'(\mathbf{R})^0$-equivariant, so the proper discontinuity of the left Γ'_0-action on X_0 is equivalent to the same for the left Γ'_0-action on $G'(\mathbf{R})^0/K$. But Γ'_0 is discrete in $G'(\mathbf{R})^0$ (Remark 6.3) and it is torsion-free, so Proposition 4.15 in the handout "$\mathrm{GL}_2(\mathbf{Z})$-action and modular forms" proves the proper discontinuity of its left action on $G'(\mathbf{R})^0/K$.

It remains to prove that (6.1) is a real-analytic isomorphism. The theory of reductive groups implies that the natural multiplication map $Z_G \times G' \to G$ is an isogeny (hence an isomorphism on Lie algebras, as we are in characteristic 0), so the map $Z_G(\mathbf{R})^0 \times G'(\mathbf{R})^0 \to G(\mathbf{R})^0$ between *connected* Lie groups is a finite-degree real-analytic covering map. Hence, $\pi : G'(\mathbf{R})^0 \to G(\mathbf{R})^0/Z_G(\mathbf{R})^0$ is a finite-degree real-analytic covering map. Since $Z_{h_0} \cap G(\mathbf{R})^0$ contains $Z_G(\mathbf{R})^0$ and

$$\pi^{-1}((Z_{h_0} \cap G(\mathbf{R})^0)/Z_G(\mathbf{R})^0) = Z_{h_0} \cap G'(\mathbf{R})^0$$

due to the definition of Z_{h_0} as a centralizer, the right side contains $\ker \pi$ and induces an isomorphism

$$(Z_{h_0} \cap G'(\mathbf{R})^0)/(\ker \pi) \simeq (Z_{h_0} \cap G(\mathbf{R})^0)/Z_G(\mathbf{R})^0.$$

Passing to the quotient by $Z_{h_0} \cap G'(\mathbf{R})^0$ on the source of π and the quotient by $(Z_{h_0} \cap G(\mathbf{R})^0)/Z_G(\mathbf{R})^0$ on the target of π therefore yields an isomorphism. But this isomorphism is precisely the map (6.1). ■

The method of proof of the preceding proposition yields a technically useful result, as follows. Let (G, X) be a Shimura datum, and let $G^{\mathrm{ad}} = G/Z_G$ denote the quotient of G modulo its center (the "adjoint quotient"; e.g., $\mathrm{GL}_2^{\mathrm{ad}} = \mathrm{PGL}_2$). This quotient is a connected semisimple $\mathbf{Q}$-group, its center is trivial, and the natural map $G' \to G^{\mathrm{ad}}$ from the derived group G' of G^{ad} is an isogeny whose kernel is the finite center $Z_{G'}$ of G'. In particular, $G'(\mathbf{R})^0 \to G^{\mathrm{ad}}(\mathbf{R})^0$ is a finite-degree real-analytic covering space (so the formation of images and preimages under this map induces a bijective correspondence between maximal compact subgroups).

Corollary 6.7. *The $(G^{\mathrm{ad}})(\mathbf{R})$-orbits $\{X_j\}$ in X are unions of connected components, and each pair (G^{ad}, X_j) is a Shimura datum.*

Proof. Since $G(\mathbf{R})^0 \to G^{\mathrm{ad}}(\mathbf{R})^0$ is surjective, the isomorphism (6.1) shows that $(G^{\mathrm{ad}})^0(\mathbf{R})$ acts transitively on each connected component of X. This gives the claim concerning $G^{\mathrm{ad}}(\mathbf{R})$-orbits on X, and the verification of Axioms I and II is then straightforward. Since G^{ad} has trivial center, the final condition in the definition of a Shimura datum is immediate. ■

The importance of Corollary 6.7 is that it reduces many problems for Shimura data to the case when G is an adjoint semisimple group. (Note that for a general connected reductive $\mathbf{Q}$-group G, if Γ is an arithmetic subgroup of $G'(\mathbf{Q})$ then its image in $G^{\mathrm{ad}}(\mathbf{Q})$ is an arithmetic subgroup since $G' \to G^{\mathrm{ad}}$ is an isogeny.) For instance, in the case of a Shimura datum for GL_n, we can pass to the group PGL_n.

7. Complex structure and canonical models

We now consider a pair (G, X) over $\mathbf{R}$ that satisfies Axioms I and II, with G also assumed to be reductive. The main aim of this section is to establish the existence of a canonically associated complex-analytic structure on X, generalizing the construction given for our GL_2-example in Example 2.1.

For any $h \in X$ the $h(\mathbf{C}^\times)$-action on X fixes h, and the induced action on $\mathrm{T}_h(X)$ makes $h(\pm i)$ act as negation (Remark 5.7), so the elements $h(e^{\pm 2\pi i/8})$ act as automorphisms $J_{h,\pm i}$ whose square is negation. Moreover, these actions are negative to each other (since $e^{-2\pi i/8} = -ie^{2\pi i/8}$) and the left translation by g intertwines $J_{h,\pm i}$ at h with $J_{g.h,\pm i}$ at $g.h$ since $g \cdot h(z) = (g.h)(z) \cdot g$ in G for all $h \in X$, $g \in G(\mathbf{R})$, and $z \in \mathbf{C}^\times$. In view of how the C^∞-structure on X is defined as a $G(\mathbf{R})$-coset space, it follows that for each $i = \sqrt{-1} \in \mathbf{C}$ the automorphism $J_{h,i}$ has C^∞-dependence on h. Thus, upon fixing a choice of i we get a $G(\mathbf{R})$-invariant almost complex structure J on X by defining J_h to be the $h(e^{2\pi i/8})$-action on $\mathrm{T}_h(X)$ for all $h \in X$.

Theorem 7.1. *The $G(\mathbf{R})$-invariant almost complex structure J on X is integrable.*

The complex structure arising from J is characterized by the property that for all representations $\rho : G \to \mathrm{GL}(U)$ over $\mathbf{R}$, the Hodge filtrations $\{F^p_{\rho,h}\}$ on the fibers of the trivial holomorphic vector bundle $U_{\mathbf{C}} \times X$ over X vary holomorphically in h. (That is, there is a decreasing chain of holomorphic subbundles $\mathscr{F}^p_\rho$ of the vector bundle $X \times U_{\mathbf{C}}$ over X such that the h-fiber of the chain is $\{F^p_{\rho,h}\}$.)

Moreover, this holomorphic filtration property for one faithful ρ determines the complex structure on X.

The final part of the theorem is useful in practice when identifying the holomorphic structure in explicit examples. For instance, consider the $\mathbf{Q}$-group $G = \mathrm{GSp}_{2g}$ and the set X of pure Hodge structures on $\mathbf{R}^{2g}$ of weight -1 and type $\{(-1, 0), (0, -1)\}$ corresponding to complex structure on $\mathrm{H}_1(A, \mathbf{R}) \simeq \mathbf{R}^{2g}$ for principally polarized complex-analytic abelian varieties (A, ψ) equipped with a trivialization $\mathrm{H}_1(A, \mathbf{Z}) \simeq \mathbf{Z}^{2g}$ making the principal polarization

$$\psi : \mathrm{H}_1(A, \mathbf{Z}) \times \mathrm{H}_1(A, \mathbf{Z}) \to \mathbf{Z}(1)$$

have the matrix $\pm\left(\begin{smallmatrix} 0 & 2\pi i \cdot 1_g \\ -2\pi i \cdot 1_g & 0 \end{smallmatrix}\right)$. For $g = 1$ this recovers our GL_2-example (since all elliptic curves are uniquely principally polarized). The complex-analytic theory of principally polarized abelian varieties ensures that X is a $G(\mathbf{R})$-conjugacy class of algebraic homomorphisms $\mathbf{C}^\times \to G(\mathbf{R}) \subseteq \mathrm{Aut}(\mathbf{R}^{2g})$ making (G, X) a Shimura datum, and the canonical faithful representation $\rho : G_{\mathbf{R}} \to \mathrm{GL}(\mathbf{R}^{2g})$ can be used to identify

X equipped with its complex structure via Theorem 7.1 as the disjoint union of the two classical Siegel half-spaces (one for each $i = \sqrt{-1} \in \mathbf{C}$).

Proof. We will first prove integrability of the almost complex structure by using comparison with a $G(\mathbf{R})$-equivariant complex structure obtained by pullback from one on a flag variety quotient of $G(\mathbf{C})$ (generalizing the GL_2-case, which rests on the flag variety $\mathbf{CP}^1 = \mathrm{GL}_2(\mathbf{C})/B(\mathbf{C})$ for Borel $\mathbf{C}$-subgroups B in GL_2 *not* defined over $\mathbf{R}$). Then we will use related ideas to verify the proposed characterization in terms of algebraic representations of G. Throughout the argument, we will use facts from the theory of reductive groups.

The entire problem is local on X in an evident sense, so by Corollary 6.7 (which only requires Axioms I and II for an $\mathbf{R}$-group, not the $\mathbf{Q}$-structure) we may replace G with G/Z_G to arrange that G is semisimple. Hence, by Axiom II, for each $h \in X$ the centralizer K_h of h in $G(\mathbf{R})$ has identity component that is a maximal *connected* compact subgroup of $G(\mathbf{R})$.

Fix a choice of $h \in X$, and consider the integrability problem. Our task is to study the almost complex structure on $G(\mathbf{R})/K_h$ defined by using the left translation action of $(g.h)(e^{2\pi i/8})$ at the coset of each $g \in G(\mathbf{R})$. It makes sense to pose the same problem for $G(\mathbf{R})/K_h^0$, as then we just need to pass to the quotient by the properly discontinuous right action of the finite group K_h/K_h^0 (with respect to which the almost complex structure is invariant!) to recover our claims for $G(\mathbf{R})/K_h$. By Proposition 5.6, K_h^0 is the identity component of the $G(\mathbf{R})$-centralizer of $h(i)$. In other words, $\mathfrak{k} := \mathrm{Lie}(K_h^0) = \mathrm{Lie}(K_h)$ is the 1-eigenspace of the involution $\mathrm{Ad}(h(i)) \in \mathrm{Aut}(\mathfrak{g})$ ($\mathfrak{g} = \mathrm{Lie}(G(\mathbf{R})) = \mathrm{Lie}(G)$), so the -1-eigenspace $\mathfrak{p}$ in $\mathfrak{g}$ maps isomorphically onto $\mathrm{Lie}(G(\mathbf{R})/K_h^0)$, and is equipped with a complex structure via the adjoint action of $h(e^{2\pi i/8})$ (which preserves $\mathfrak{p}$ since $h(\mathbf{C}^\times)$ is commutative). The adjoint action of $h(i)$ on $[\mathfrak{p}, \mathfrak{k}]$ is via negation, so $[\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}$. Similarly, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$.

Consider the complex Lie group $G(\mathbf{C})$. This is connected since the $\mathbf{C}$-group $G_{\mathbf{C}}$ is connected (it is a nontrivial consequence of Serre's GAGA theorems that for a connected $\mathbf{C}$-scheme of finite type, the $\mathbf{C}$-points are connected in the analytic topology). Its Lie algebra is

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}_{\mathbf{C}} = \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}_{\mathbf{C}}^{i} \oplus \mathfrak{p}_{\mathbf{C}}^{-i}$$

where $\mathfrak{p}_{\mathbf{C}}^{\pm i}$ is the $\pm i$-eigenspace in $\mathfrak{p}_{\mathbf{C}}$ for the adjoint action of $h(e^{2\pi i/8})$ on $\mathfrak{p}$. In terms of the Hodge structure on $\mathfrak{g}$ associated to $\mathrm{Ad}_{G(\mathbf{R})} \circ h$, we have

$$\mathfrak{g}_{\mathbf{C}}^{0,0} = \mathfrak{k}_{\mathbf{C}}, \quad \mathfrak{g}_{\mathbf{C}}^{1,-1} = \mathfrak{p}_{\mathbf{C}}^{-i}, \quad \mathfrak{g}_{\mathbf{C}}^{-1,1} = \mathfrak{p}_{\mathbf{C}}^{i}$$

since $z^{-\epsilon}\overline{z}^{\epsilon} = z^{-2\epsilon}$ for $z \in S^1$ and $\epsilon = \pm 1$. It follows from the compatibility of Lie bracket and the adjoint action of $G(\mathbf{C})$ on $\mathfrak{g}_{\mathbf{C}}$ that

$$[\mathfrak{k}_{\mathbf{C}}, \mathfrak{p}_{\mathbf{C}}^{-i}] \subseteq \mathfrak{p}_{\mathbf{C}}^{-i}, \quad [\mathfrak{p}_{\mathbf{C}}^{-i}, \mathfrak{p}_{\mathbf{C}}^{-i}] = 0$$

(the latter because the adjoint action of $h(e^{2\pi i/8})$ has vanishing -1-eigenspace). Thus, $\mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}_{\mathbf{C}}^{-i}$ is a Lie subalgebra of $\mathfrak{g}_{\mathbf{C}}$. It therefore exponentiates to a connected complex Lie subgroup H of $G(\mathbf{C})$, but is H *closed*? Even better:

Lemma 7.2. *The Lie subgroup H in $G(\mathbf{C})$ is Zariski-closed.*

In the GL_2-case, H is a Borel subgroup not defined over $\mathbf{R}$ (arising as the $\mathrm{GL}_2(\mathbf{C})$-stabilizer of a $\mathbf{C}$-line L in $\mathbf{C}^2$ not defined over $\mathbf{R}$; such $\mathbf{C}$-lines correspond bijectively to complex structures on $\mathbf{R}^2$ via $\mathbf{R}^2 \simeq \mathbf{C}^2/L$, as in Example A.1).

Proof. Pick a faithful linear representation $\rho : G \to \mathrm{GL}(U)$ over $\mathbf{R}$, and let $\{F_h^p\}$ be the Hodge filtration on $U_{\mathbf{C}}$ associated to $\rho_{\mathbf{R}} \circ h$ (where $\rho_{\mathbf{R}}$ denotes ρ restricted to $\mathbf{R}$-points). Define $d(p) = \dim F_h^p$ and let $Y = \mathrm{Flag}_d(U_{\mathbf{C}})$ be the associated flag variety (classifying descending flags in $U_{\mathbf{C}}$ whose pth stage has dimension $d(p)$). Using $\rho_{\mathbf{C}}$, we get an algebraic orbit map $f : G_{\mathbf{C}} \to Y$ through $F_h^\bullet$ (i.e., $f(g) = \rho(g)_{\mathbf{C}}(F_h^\bullet)$). The tangent map at the identity has the form

$$\mathrm{d}f(1) : \mathfrak{g}_{\mathbf{C}} \to \mathrm{End}(U_{\mathbf{C}})/F^0(\mathrm{End}(U_{\mathbf{C}}))$$

by the arguments used in the proof of Proposition 5.3. (Here we use the Hodge structure on $\mathrm{End}(U)$ arising from the one on U via $\rho_{\mathbf{R}} \circ h$.)

Relative to the Hodge structure on $\mathfrak{g}$ via $\mathrm{Ad}_{G(\mathbf{R})} \circ h$, the subspace

$$F^0(\mathfrak{g}_{\mathbf{C}}) = \mathfrak{g}_{\mathbf{C}}^{0,0} \oplus \mathfrak{g}_{\mathbf{C}}^{1,-1} = \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}_{\mathbf{C}}^{-i} = \mathrm{Lie}(H)$$

is the kernel of $\mathrm{d}f(1)$ because the map $\mathrm{d}f(1)$ is $\mathbf{C}^\times$-equivariant (due to the same calculations as in the proof of Proposition 5.3). But thinking algebraically, the stabilizer of $G_{\mathbf{C}}$ at $F_h^\bullet$ is a Zariski-closed $\mathbf{C}$-subgroup that is the f-fiber over $F_h^\bullet$, so its Lie algebra is $\ker \mathrm{d}f(1) = \mathrm{Lie}(H)$. This proves that H is the identity component of the $\mathbf{C}$-points of the stabilizer $\mathrm{Stab}_{G_{\mathbf{C}}}(F_h^\bullet)$, so H arises from the algebraic identity component $\mathrm{Stab}_{G_{\mathbf{C}}}(F_h^\bullet)^0$ (since connectedness is preserved under passage to $\mathbf{C}$-points). ■

Example 7.3. When working with the Shimura datum for g-dimensional principally polarized abelian varieties, the subgroup H is a so-called Siegel parabolic subgroup of GSp_{2g} (a Borel subgroup of GL_2 when $g = 1$). In terms of $g \times g$ block matrices, it is conjugate to the group of matrices $\left(\begin{smallmatrix} a & b \\ 0 & (a^{\mathrm{t}})^{-1} \end{smallmatrix}\right)$ such that $ab(a^{\mathrm{t}})^{-1} = b$, with unipotent radical U consisting of matrices $\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ for symmetric u. By inspection, U is commutative. Likewise, in general the commutative subalgebra $\mathfrak{p}_{\mathbf{C}}^{-i}$ of $\mathrm{Lie}(H)$ is the Lie algebra of the unipotent radical U of H, so this U is commutative (in contrast with the unipotent radicals of smaller parabolic subgroups, such as the Borel subgroups, when $g > 1$).

Consider the natural map $\xi : G(\mathbf{R})/K_h^0 \to G(\mathbf{C})/H = \mathrm{Flag}_d(U_{\mathbf{C}})$ (with $d : \mathbf{Z} \to \mathbf{Z}_{\geq 0}$ as in the proof of the Lemma 7.2). The induced map on tangent spaces at the identity cosets is

$$\mathfrak{p} = \mathfrak{g}/\mathfrak{k} \to \mathfrak{g}_{\mathbf{C}}/F^0(\mathfrak{g}_{\mathbf{C}}) = \mathfrak{g}_{\mathbf{C}}^{1,-1} = \mathfrak{p}_{\mathbf{C}}^{i}.$$

This is an isomorphism due to how $\mathfrak{p}_{\mathbf{C}}^{i}$ is constructed using the complex structure on $\mathfrak{p}$ defined by the adjoint action on $h(e^{2\pi i/8})$ (also see Example A.1). It follows that ξ is a local real-analytic isomorphism near the identity, so by $G(\mathbf{R})$-equivariance the map ξ is a local real-analytic isomorphism onto an open image. Hence, K_h^0 must be the identity component of $H' := H \cap G(\mathbf{R})$, and ξ is the composition of the quotient map $G(\mathbf{R})/K_h^0 \to G(\mathbf{R})/H'$ and a real-analytic isomorphism of $G(\mathbf{R})/H'$ onto an open subset of $G(\mathbf{C})/H$. This open embedding gives $G(\mathbf{R})/H'$ a complex structure, and the complex structure is $G(\mathbf{R})$-invariant since the left $G(\mathbf{R})$-action on $G(\mathbf{R})/H'$ is compatible (via ξ) with the holomorphic left $G(\mathbf{C})$-action on $G(\mathbf{C})/H$.

We claim that H' arises as the $\mathbf{R}$-points of a closed $\mathbf{R}$-subgroup of G. Note that the map $G(\mathbf{R}) \to G(\mathbf{C})$ of real Lie groups is the map on $\mathbf{R}$-points induced by the natural inclusion map of $\mathbf{R}$-groups $G \to \mathrm{Res}_{\mathbf{C}/\mathbf{R}}(G_{\mathbf{C}})$. Lemma 7.2 provides a closed $\mathbf{C}$-subgroup $H^{\mathrm{alg}} \subseteq G(\mathbf{C})$ such that $H = H^{\mathrm{alg}}(\mathbf{C})$ inside of $G(\mathbf{C})$, so H' is the group of $\mathbf{R}$-points of $G \cap \mathrm{Res}_{\mathbf{C}/\mathbf{R}}(H^{\mathrm{alg}})$.

It follows from the algebraicity of H' inside of $G(\mathbf{R})$ that H' has *finite* component group, so $G(\mathbf{R})/K_h^0$ is a Galois finite-degree real-analytic covering space of $G(\mathbf{R})/H'$, with covering group H'/K_h^0 acting on the right. Thus, $G(\mathbf{R})/K_h^0$ inherits the complex structure from $G(\mathbf{R})/H'$, and this is invariant for the left $G(\mathbf{R})$-action. But at the identity coset of $G(\mathbf{R})/H'$ the almost complex structure on the tangent space $\mathfrak{p}$ is that of $G(\mathbf{C})/H$, which is $\mathfrak{p}_{\mathbf{C}}^{i}$. By construction, this is the adjoint action of $h(e^{2\pi i/8})$, which coincides with the derivative (at the identity coset) of the left-translation action by $h(e^{2\pi i/8})$. The same then holds for the almost complex structure just constructed on $G(\mathbf{R})/K_h^0$. But our initial almost complex structure of interest on $G(\mathbf{R})/K_h^0$ is also invariant with respect to the left $G(\mathbf{R})$-action, and at the identity coset it coincides with the left-translation action by $h(e^{2\pi i/8})$, so these almost complex structures coincide. This solves our integrability problem for $G(\mathbf{R})/K_h^0$, and at the outset we saw that we may then pass to the quotient by the right action of K_h/K_h^0 to get the desired complex structure on $G(R)/K_h = X$. This completes the proof of integrability of the initial almost complex structure on X.

The preceding method of proof gives us more: it shows that for any single faithful representation $\rho : G \to \mathrm{GL}(U)$, the desired holomorphicity property holds (since holomorphicity of a C^∞-map between complex manifolds is equivalent to the compatibility of almost complex structures). Moreover, it shows that this property for a single such ρ uniquely characterizes the complex structure. Finally, a general ρ embeds into a faithful one (by forming a direct sum against a faithful representation), so a functoriality argument establishes the desired holomorphicity property for any (ρ, U). ■

Now let (G, X) be a Shimura datum (so G is a $\mathbf{Q}$-group, not an $\mathbf{R}$-group). Let Γ be an arithmetic subgroup of $G(\mathbf{Q})$. By Example 6.4 and Proposition 6.6, there is a finite-index normal subgroup Γ' in Γ whose action on X is properly discontinuous. Thus, the quotient $\Gamma' \backslash X$ makes sense as a complex manifold. This is equipped with a left action by the *finite* group Γ/Γ', and an analytic quotient by this finite group action is the same as a quotient $\Gamma \backslash X$. Hence, in general (when Γ is not assumed to be torsion-free), the Γ-action on X is discontinuous and the singularities on the quotient $\Gamma \backslash X$ (a normal complex-analytic space) are not too bad. In particular, away from a nowhere-dense analytic set in $\Gamma \backslash X$ the analytic structure is that of a complex manifold.

Work of Baily–Borel (using the normality of the complex-analytic space $\Gamma \backslash X$, and the construction of many Γ-automorphic forms on $G(\mathbf{R})$) proves that $\Gamma \backslash X$ always admits a structure of quasi-projective normal algebraic variety over $\mathbf{C}$. Their method also yields a uniqueness result for this algebraic structure, as well as a functorial property for it (despite the inapplicability of Serre's GAGA theorem when $\Gamma \backslash X$ is non-compact; the functorial aspect of the uniqueness is not as strong as in GAGA). For instance, in our GL_2-example this is the usual algebraic structure on classical analytic modular curves (whose uniqueness can be proved directly, using the big Picard theorem).

The miracle discovered by Shimura in many cases (and systematized by Deligne) is that these quasi-projective algebraic varieties over $\mathbf{C}$ admit a (necessarily unique) descent to $\overline{\mathbf{Q}}$, and more importantly even admit a uniquely characterized descent to a specific number field (the *reflex field* $E(G, X)$ associated to (G, X) in accordance with a procedure inspired by the special case $G = \mathrm{GSp}_{2g}$ and the Main Theorem of Complex Multplication for abelian varieties). This descent is called the *canonical model* of $\Gamma \backslash X$. At the time of Deligne's work on the problem, he was able to construct canonical models in many cases (going beyond those considered by Shimura), relying on Mumford's work on moduli of abelian varieties to handle GSp_{2g} and then algebraic group methods to bootstrap to other cases.

The general proof of the existence of canonical models for *any* Shimura datum involved the work of many mathematicians over many years. The resulting "arithmetic structure" on the quotients $\Gamma \backslash X$ underlies much of the arithmetic theory of automorphic forms. However, this arithmetic structure is characterized by an entirely Galois-theoretic criterion (no moduli problems!), and the Galois-theoretic viewpoint is not well-suited to integrality questions. Integrality issues naturally arise in the study of Galois actions on the étale cohomology of canonical models (where one wants to relate Frobenius actions to Hecke operators, vastly generalizing the Eichler–Shimura relation in the case of modular curves). In a special class of cases (PEL Shimura varieties) one can introduce integral moduli problems, and for our GL_2 example the work of Katz–Mazur and Deligne–Rapoport on moduli of (generalized) elliptic curves gives a definitive theory over $\mathbf{Z}$. In general the study of integral structures on canonical models of Shimura varieties remains a source of many difficult open questions (though in recent years there has been a lot of progress).

Appendix A. Some tangent space formalism

Let M be a complex manifold, and $M_{\mathbf{R}}$ the underlying smooth manifold. This appendix describes the conceptual procedure to identify $\mathrm{T}_m(M_{\mathbf{R}})$ as the underlying $\mathbf{R}$-vector space of $\mathrm{T}_m(M)$. We will leave it to the reader to check that this is functorial: if $f : M' \to M$ is a holomorphic map and $f_{\mathbf{R}} : M'_{\mathbf{R}} \to M_{\mathbf{R}}$ is the underlying C^∞ map then $\mathrm{d}f_{\mathbf{R}}(m)$ is identified with the $\mathbf{R}$-linear map underlying the $\mathbf{C}$-linear map $\mathrm{d}f(m)$.

The intrinsic relationship between the two notions of tangent space stems from the fact that local $\mathbf{C}$-valued C^∞ functions on $M_{\mathbf{R}}$, such as local holomorphic functions on M, are $\mathbf{C}$-linear combinations of local $\mathbf{R}$-valued C^∞-functions. This enables local C^∞ vector fields on $M_{\mathbf{R}}$ to act on such functions via scalar extension from $\mathbf{R}$ to $\mathbf{C}$.

The natural $\mathbf{R}$-linear map $c_m : \mathrm{T}_m(M_{\mathbf{R}}) \to \mathrm{T}_m(M)$ is defined as follows. For any point-derivation $\partial \in \mathrm{T}_m(M_{\mathbf{R}})$ at m (viewed as an $\mathbf{R}$-linear map $\mathscr{O}_{M_{\mathbf{R}},m} \to \mathbf{R}$ satisfying the Leibnitz rule at m and killing germs that vanish to first order) and holomorphic germ $f = u + iv$ for smooth $\mathbf{R}$-valued germs u and v, consider $c_m(\partial) : f \mapsto \partial(u) + i\partial(v) \in \mathbf{C}$. This is $\mathbf{C}$-linear in f, satisfies the Leibnitz rule, and kills such f that vanish to first order at m in the holomorphic sense (as that forces u and v to vanish to first order at m in the C^∞ sense), so $c_m(\partial) \in \mathrm{T}_m(M)$. It is clear that c_m is independent of the choice of i (as negating i

also negates v), and in terms of local holomorphic coordinates $\{z_j = x_j + iy_j\}$ we have

$$c_m(\partial_{x_j}|_m) = \partial_{z_j}|_m, \;\; c_m(\partial_{y_j}|_m) = i\partial_{z_j}|_m.$$

Hence, the **R**-linear c_m is surjective, so it is an **R**-linear isomorphism for dimension reasons. Via the identification c_m of $\mathrm{T}_m(M_{\mathbf{R}})$ with $\mathrm{T}_m(M)$, the **C**-linear structure on $\mathrm{T}_m(M_{\mathbf{R}})$ is readily computed: since $c_m(\partial_{x_j}|_m) = \partial_{z_j}|_m$ and $i\partial_{z_j}|_m = c_m(\partial_{y_j}|_m)$, we have $i\partial_{x_j}|_m = \partial_{y_j}|_m$ in $\mathrm{T}_m(M_{\mathbf{R}})$ equipped with its **C**-linear structure via c_m, exactly as we would expect.

Let's push this a step further. The **C**-linearized map $\xi_m : \mathrm{T}_m(M_{\mathbf{R}})_{\mathbf{C}} \to \mathrm{T}_m(M)$ obtained from the **R**-linear isomorphism c_m is surjective and satisfies

$$\partial_{x_j}|_m + \frac{1}{i} \otimes \partial_{y_j}|_m \mapsto 2\partial_{z_j}|_m, \;\; \partial_{x_j}|_m - \frac{1}{i} \otimes \partial_{y_j}|_m \mapsto 0.$$

Thus, the complex conjugation operator $c \otimes v \mapsto \overline{c} \otimes v$ on $\mathrm{T}_m(M_{\mathbf{R}})_{\mathbf{C}}$ (denoted $w \mapsto \overline{w}$) moves $\ker \xi$ to a complementary **C**-subspace $\overline{\ker \xi}$ (the scalar extension of $\ker \xi$ along complex conjugation). This complement therefore maps isomorphically onto $\mathrm{T}_m(M)$ and thereby identifies $\mathrm{T}_m(M)$ as a *subspace* of $\mathrm{T}_m(M_{\mathbf{R}})_{\mathbf{C}}$:

(A.1) $$\mathrm{T}_m(M_{\mathbf{R}})_{\mathbf{C}} = \mathrm{T}_m(M) \oplus \overline{\mathrm{T}_m(M)}.$$

(Compare this with Example (A.1).) In terms of this decomposition, we have

$$\frac{1}{2}\left(\partial_{x_j}|_m + \frac{1}{i} \otimes \partial_{y_j}|_m\right) = \partial_{z_j}|_m, \;\; \frac{1}{2}\left(\partial_{x_j} - \frac{1}{i} \otimes \partial_{y_j}|_m\right) = \overline{\partial}_{z_j}|_m,$$

the so-called "Wirtinger formulas" (except that traditionally $\overline{\partial}_{z_j}$ is denoted $\partial_{\overline{z}_j}$, and the $\otimes$ is suppressed). (Don't confuse the fact that $i \otimes \partial_{x_j}|_m \neq \partial_{y_j}|_m$ in $\mathrm{T}_m(M_{\mathbf{R}})_{\mathbf{C}}$ with the fact that the **C**-linear structure on $\mathrm{T}_m(M_{\mathbf{R}})$ makes $i\partial_{x_j}|_m = \partial_{y_j}|_m$.)

If we dualize the **C**-linear identification $\mathrm{T}_m(M_{\mathbf{R}})_{\mathbf{C}} = \mathrm{T}_m(M) \oplus \overline{\mathrm{T}_m(M)}$ then we get

$$\mathrm{Cot}_m(M_{\mathbf{R}})_{\mathbf{C}} = \mathrm{Cot}_m(M) \oplus \overline{\mathrm{Cot}_m(M)}$$

since dualizing and extension of scalars commute. (Concretely, this says that the complex conjugation operator on $\mathrm{Cot}_m(M_{\mathbf{R}})_{\mathbf{C}}$ carries $\mathrm{Cot}_m(M)$ to a **C**-linear complement.) In terms of this decomposition, by computing dual bases to the **C**-basis $\{\partial_{z_j}|_m, \overline{\partial}_{z_j}|_m\}$ we get

$$\mathrm{d}z_j(m) = \mathrm{d}x_j(m) + i \otimes \mathrm{d}y_j(m), \;\; \overline{\mathrm{d}z_j(m)} = \mathrm{d}x_j(m) - i \otimes \mathrm{d}y_j(m)$$

in $\mathrm{Cot}_m(M_{\mathbf{R}})_{\mathbf{C}}$. Traditionally the $\otimes$ is suppressed, thereby identifying $\overline{\mathrm{d}z_j(m)}$ with $\mathrm{d}\overline{z}_j(m)$ in terms of the d-operator on smooth **C**-valued functions on $M_{\mathbf{R}}$ (such as $\overline{z}_j$).