Math 248B. Applications of base change for coherent cohomology

1. Motivation

Recall the following fundamental general theorem, the so-called "cohomology and base change" theorem:

Theorem 1.1 (Grothendieck)**.** *Let $f : X \to S$ be a proper morphism of schemes with S locally noetherian, and let $\mathscr{F}$ be an S-flat coherent sheaf on X. For $i \geq 0$ and $s \in S$, assume that the natural base change morphism $\varphi_s^i : \mathrm{R}^i f_*(\mathscr{F})_s \otimes_{\mathscr{O}_{S,s}} k(s) \to \mathrm{H}^i(X_s, \mathscr{F}_s)$ is surjective. Then $\varphi_{s'}^i$ is an isomorphism for s' near s and the following are equivalent:*

(1) *φ_s^{i-1} is surjective*
(2) *the finite $\mathscr{O}_{S,s}$-module $\mathrm{R}^i f_*(\mathscr{F})_s$ is free.*

This result is proved in §12 of Chapter III of Hartshorne's *Algebraic Geometry* under a projectivity assumption on $X \to S$. But this projectivity is hardly used in the proof: its only purpose is to guarantee coherence of higher direct images, which is proved more generally for proper morphisms in EGA III_1, 3.2.1 by a clever deduction from the projective case via Chow's Lemma and spectral sequences.

A nice discussion of this theorem is given in §5 of Chapter II of Mumford's "Abelian varieties" (granting of course the coherence of higher direct images under a proper morphism). Vakil's notes on this topic are also illuminating.

In this handout, we wish to deduce some general consequences of the "cohomology and base change" theorem (e.g., explain how it really says something about general base change, not just passage to fibers). To give a feeling for the power of Grothendieck's theorem, we now record some important consequences:

Corollary 1.2. *If $\mathrm{H}^i(X_s, \mathscr{F}_s) = 0$ for some $s \in S$ then* (i) *the same holds for all s' near s,* (ii) *$\mathrm{R}^i f_*(\mathscr{F})$ vanishes near s, and* (iii) *$\varphi_{s'}^{i-1}$ is an isomorphism for s' near s.*

In the case $i = 1$, $f_*\mathscr{F}$ is locally free near s and $\varphi_{s'}^0 : f_*(\mathscr{F})_{s'} \otimes_{\mathscr{O}_{S,s'}} k(s') \to \mathrm{H}^0(X_{s'}, \mathscr{F}_{s'})$ is an isomorphism for all s' near s.

Proof. Obviously φ_s^i is surjective, so by Theorem 1.1 this map is an isomorphism. Hence, $\mathrm{R}^i f_*(\mathscr{F})_s$ vanishes by Nakayama's Lemma. It follows that $\mathrm{R}^i f_*(\mathscr{F})$ vanishes near s, so it is locally free (of rank 0). Thus, by Theorem 1.1(2) , $\varphi_{s'}^{i-1}$ is surjective – and therefore an isomorphism – for all s' near s. Also, since $\varphi_{s'}^i$ is an isomorphism for s' near s, we deduce the vanishing of $\mathrm{H}^i(X_{s'}, \mathscr{F}_{s'})$ for s' near s.

Now suppose $i = 1$. In this case $\varphi_{s'}^0$ is an isomorphism for s' near s, but trivially $\varphi_{s'}^{-1}$ is surjective for all s'. Hence, by Theorem 1.1 with $i = 0$, the $\mathscr{O}_{S,s}$-module $(f_*\mathscr{F})_s$ is free. Thus, $f_*\mathscr{F}$ is locally free near s. ■

The following result is especially useful.

Corollary 1.3. *Let $f : X \to S$ be a proper surjective flat map whose geometric fibers are reduced and connected. Then the natural map $\mathscr{O}_S \to f_*\mathscr{O}_X$ is an isomorphism.*

Proof. For any $s \in S$, the $k(s)$-algebra of global functions $\mathrm{H}^0(X_s, \mathscr{O}_{X_s})$ is nonzero and finite-dimensional over $k(s)$ since X_s is proper and non-empty, and its formation commutes with any extension on $k(s)$ (by flatness of field extensions). Passing to a geometric fiber therefore gives the algebra of global functions on a reduced connected proper scheme over an algebraically closed field K, which must coincide with K since it is reduced and has no nontrivial idempotents (by connectedness). Thus, $\mathrm{H}^0(X_s, \mathscr{O}_{X_s})$ is 1-dimensional over $k(s)$, so the natural injective map $k(s) \to \mathrm{H}^0(X_s, \mathscr{O}_{X_s})$ is an isomorphism.

Since X is S-flat, Theorem 1.1 can be applied with $\mathscr{F} = \mathscr{O}_X$. Consider the base change morphism

$$\varphi_s^0 : f_*(\mathscr{O}_X)_s \otimes_{\mathscr{O}_{S,s}} k(s) \to \mathrm{H}^0(X_s, \mathscr{O}_{X_s}) = k(s).$$

This is nonzero, since $1 \mapsto 1$, so it is surjective. Thus, it is an isomorphism. But φ_s^{-1} is trivially surjective, so $f_*(\mathscr{O}_X)_s$ is free, necessarily of rank 1 since its fiber modulo $\mathfrak{m}_s$ is identified with $k(s)$ via φ_s^0. It follows that the coherent $\mathscr{O}_S$-module $f_*\mathscr{O}_X$ is locally free of rank 1, so the map $\mathscr{O}_S \to f_*\mathscr{O}_X$ of sheaves of algebras must be an isomorphism because modulo $\mathfrak{m}_s$ it becomes the structural morphism $k(s) \to f_*(\mathscr{O}_X)_s \otimes_{\mathscr{O}_{S,s}} k(s)$ that we have seen is an isomorphism. ■

Remark 1.4. A special case Theorem 1.1 is seen by taking $i = d$ when f has all fibers of dimension $\leq d$. In this case φ_s^{d+1} is certainly surjective. But $\mathrm{R}^{d+1}f_*(\mathscr{F}) = 0$ (and hence is locally free) because the theorem on formal functions identifies the completed stalk at $s \in S$ with the inverse limit of cohomologies $\mathrm{H}^{d+1}(X_{s,n}, \mathscr{F}_{s,n})$ of the infinitesimal fibers, all of which vanish (because $X_{s,n}$ is noetherian of dimension $\leq d$). Hence, Theorem 1.1 gives that φ_s^d is surjective (and hence an isomorphism) for all $s \in S$. But this surjectivity can be seen very directly by replacing S with $\mathrm{Spec}(\mathscr{O}_{S,s})$ and identifying φ_s^d with the map induced by applying $\mathrm{R}^d f_*$ to the surjection $\mathscr{F} \to \mathscr{F}_s$ of coherent $\mathscr{O}_X$-modules (viewing X_s as a closed subscheme of X). The point is that in this "localized" setting, $\mathrm{R}^d f_*$ is right-exact on coherent sheaves precisely because $\mathrm{R}^{d+1} f_*$ vanishes on coherent sheaves (due to the argument with the theorem on formal functions as we just explained).

For the reader who learned cohomology from Hartshorne's textbook on algebraic geometry, there may be some concern that the theorem on formal function was proved there using projective methods all over the place. In fact, the theorem is true with proper morphisms in general, and the proof in EGA III_1 is entirely different: there is no reduction to the projective case via Chow's Lemma, but rather a very clever argument due to Serre which is ultimately more illuminating than the projective method in Hartshorne.

2. Base change

Now consider the general setup in Theorem 1.1: a proper map $f : X \to S$ to a locally noetherian scheme S, and an S-flat coherent sheaf $\mathscr{F}$ on X. We want to use the fibral base change morphisms φ_s^i for $s \in S$ to study more general base change morphisms.

Proposition 2.1. *Assume φ_s^i is an isomorphism for all $s \in S$, and that φ_s^{i-1} is also an isomorphism for all $s \in S$ (or equivalently, that $\mathrm{R}^i f_*(\mathscr{F})$ is locally free on S). Consider a locally noetherian S-scheme S', the resulting cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\;q\;} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[\;p\;]{} & S \end{array}$$

and the S'-flat coherent sheaf $\mathscr{F}' = q^*\mathscr{F}$ on X'. The natural base change morphism $p^*(\mathrm{R}^i f_*\mathscr{F}) \to \mathrm{R}^i f'_*(\mathscr{F}')$ is an isomorphism.

Note that when S' is a point $s : \mathrm{Spec}(k) \to S$, the base change morphism recovers the isomorphism property of φ_s^i that is the hypothesis in the proposition. This result is easiest to understand when $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(A')$ are affine, in which case the higher direct images are associated to the cohomology modules $\mathrm{H}^i(X, \mathscr{F})$ and $\mathrm{H}^i(X', \mathscr{F}')$, and the base change morphism is the natural map

$$A' \otimes_A \mathrm{H}^i(X, \mathscr{F}) \to \mathrm{H}^i(X', \mathscr{F}')$$

arising from sheaf cohomology pullback relative to $X' \to X$. The isomorphism property for this map is especially powerful when A and A' are both artin local rings.

Proof. For clarity, lets write $\varphi_{\mathscr{F},s}^i$ rather than φ_s^i. For $s' \in S'$ over $s \in S$, the map $\varphi_{\mathscr{F}',s'}^i$ has the form

$$\mathrm{R}^i f'_*(\mathscr{F}')_{s'} \otimes_{\mathscr{O}_{S',s'}} k(s') \to \mathrm{H}^i(X'_{s'}, \mathscr{F}'_{s'}).$$

We claim that this is surjective too.

The key point is that the formation of cohomology of coherent sheaves on proper schemes over a ring commutes with *flat* scalar extension on the base ring (as one sees by identifying the pullback map in sheaf cohomology with an explicit map in Čech cohomology relative to a finite affine open cover), a main example being extension of the base field. Thus, the natural map

$$k(s') \otimes_{k(s)} \mathrm{H}^i(X_s, \mathscr{F}_s) \to \mathrm{H}^i(X'_{s'}, \mathscr{F}'_{s'}) \tag{2.1}$$

is an isomorphism. It follows that to prove the surjectivity of the $k(s')$-linear $\varphi_{\mathscr{F}',s'}^i$, it suffices to show that the image contains the $k(s)$-subspace $\mathrm{H}^i(X_s, \mathscr{F}_s)$. But it is straightforward to verify the compatibility of $\varphi_{\mathscr{F},s}^i$

and $\varphi^i_{\mathscr{F}',s'}$ with respect to the natural pullback map $\mathrm{R}^i f_*(\mathscr{F})_s \to \mathrm{R}^i f'_*(\mathscr{F}')_{s'}$ (linear over $\mathscr{O}_{S,s} \to \mathscr{O}_{S',s'}$), so the surjectivity of $\varphi^i_{\mathscr{F},s}$ implies the surjectivity of $\varphi^i_{\mathscr{F}',s'}$.

Since $\varphi^i_{\mathscr{F}',s'}$ is surjective for all $s' \in S'$, and similarly for $i-1$, by Theorem 1.1 we conclude that $\mathrm{R}^i f'_*(\mathscr{F}')$ is locally free on S'. Thus, on s'-stalks the base change morphism in the proposition goes between free $\mathscr{O}_{S',s'}$-modules of finite rank. It suffices to prove that such stalk maps are isomorphisms for all $s' \in S'$, and to prove this isomorphism property it suffices to do so modulo $\mathfrak{m}_{s'}$ (by the module-freeness). But via the isomorphisms $\varphi^i_{\mathscr{F},s}$ and $\varphi^i_{\mathscr{F}',s'}$ (and the compatibility relating them), the induced map modulo $\mathfrak{m}_{s'}$ is exactly the map (2.1) that we have seen is an isomorphism. ■

Example 2.2. Consider a genus-g S-curve $X \to S$. Taking $\mathscr{F} = \mathscr{O}_X$, the natural map φ^1_s is surjective (Remark 1.4), and φ^0_s is also surjective (as we saw in the proof of Corollary 1.3). Thus, Theorem 1.1 and Proposition 2.1 imply that $\mathrm{R}^1 f_*\mathscr{O}_X$ is a locally free sheaf on S whose formation commutes with any (locally noetherian) base change on S. In particular, the rank is g, due to φ^1_s being an isomorphism for all $s \in S$.

Note that $\Omega^1_{X/S}$ is an S-flat coherent sheaf, as it is locally free as an $\mathscr{O}_X$-module and X is S-flat. Thus, we can try to apply the base change formalism to study the higher direct images of $\Omega^1_{X/S}$. We are especially interested in $\omega_{X/S} = f_*\Omega^1_{X/S}$. We claim that this is a locally free sheaf of rank g whose formation commutes with any base change. By Proposition 2.1 with $i = 0$, it suffices to prove that $\varphi^0_{\Omega^1_{X/S},s}$ is surjective for all $s \in S$. By Theorem 1.1 and Remark 1.4, it is equivalent to prove that $\mathrm{R}^1 f_*(\Omega^1_{X/S})$ is locally free.

There are a couple of methods for proving local freeness of $\mathrm{R}^1 f_*(\Omega^1_{X/S})$, neither of which is "elementary" (and I do not know any elementary proof); this result will not be used in what follows. One method is to use an hydrogen bomb: relative Grothendieck–Serre duality provides a canonical trace isomorphism $\mathrm{R}^1 f_*(\Omega^1_{X/S}) \simeq \mathscr{O}_S$. Another method is to use an atom bomb: by the theorem on formal functions and Proposition 2.1 (applied over artin local base schemes) we can reduce to the case when the base is artin local, and then by using the main results from SGA1 on algebraization of formal deformation of curves we can reduce to the case when the base is a regular local ring (exercising care to check that this deformation and algebraization process does not lose the connectedness property for the geometric fibers). In that case the base is *reduced*, so we can apply Grauert's base change theorem (see Chapter II, §5, Corollary 2 of Mumford's "Abelian Varieties"): since the fibral cohomologies $\mathrm{H}^1(X)_s, \Omega^1_{X_s/k(s)})$ all have the same dimension (in fact, 1), reducedness of the base implies that $\mathrm{R}^1 f_*(\Omega^1_{X/S})$ is locally free.

Example 2.3. Let (E, e) be an elliptic curve over S, and for $n \geq 1$ define $\mathscr{O}(ne) = \mathscr{I}^{-n}$ for the invertible ideal sheaf $\mathscr{I}$ of the section e. We claim that $f_*(\mathscr{O}(ne))$ is a locally free sheaf of rank n whose formation commutes with any base change, and that the natural map $\mathscr{O}_S \to f_*(\mathscr{O}(e))$ is an isomorphism.

Since the genus is 1, by Riemann–Roch and Serre duality the fibral cohomologies $\mathrm{H}^1(E_s, \mathscr{O}(ne(s)))$ all vanish ($n \geq 1$) and $\mathrm{H}^0(E_s, \mathscr{O}(ne(s)))$ are n-dimensional. Thus, by Corollary 1.2 we get that $f_*(\mathscr{O}(ne))$ is locally free and its formation commutes with base change. The base change to fibers then implies that its rank is n everywhere.

To check that the natural map $\mathscr{O}_S \to f_*(\mathscr{O}(e))$ between invertible sheaves is an isomorphism, it suffices to check after passing to stalks and reducing modulo the maximal ideal. But by the base change compatibility, the resulting map is identified with the natural map $k(s) \to \mathrm{H}^0(E_s, \mathscr{O}(e(s)))$, and by the classical theory on fibers (or even geometric fibers) this is an isomorphism: it says that the only rational functions on E_s with at worst a simple pole at the origin are the constant functions.

3. Projectivity of curves

It is a classical fact that a smooth geometrically connected proper curve over a field is projective. We would like to prove a relative version of this fact. In the case of genus 1, when there is a section we proved in class that the curve is given in $\mathbf{P}^2$ by a Weierstrass cubic Zariski-locally on the base. For a genus-1 curve without a section, it is generally hopeless to prove a projectivity result, since one has no evident way to create an ample line bundle. However, for all other genera we can use the relative 1-forms to do the job:

Proposition 3.1. *Let $f : X \to S$ be a smooth proper map whose fibers are geometrically connected of dimension 1 and genus $g \neq 1$. Zariski-locally on the base, there is a projective embedding.*

Proof. If $g = 0$ then let $\mathscr{L} = (\Omega^1_{X/S})^\vee$ and if $g > 1$ then let $\mathscr{L} = (\Omega^1_{X/S})^{\otimes 3}$. By Riemann-Roch, Serre duality, and fibral degree considerations, we have $\mathrm{H}^1(X_s, \mathscr{L}_s) = 0$ for all $s \in S$. Hence, by Corollary 1.2 and Proposition 2.1 the sheaf $f_*(\mathscr{L})$ is locally free on S and its formation commutes with any base change. But $\deg(\mathscr{L}_s)$ is large enough so that by the classical theory on fibers, $\mathscr{L}_s$ is generated by its global section over X_s. Those global sections in turn are generated by the stalk $f_*(\mathscr{L})_s$ due to the surjectivity of $\varphi^0_{\mathscr{L},s}$, so it follows that over an open affine U in S the module $\mathrm{H}^0(X_U, \mathscr{L})$ of U-sections of $f_*\mathscr{L}$ generate all stalks of $\mathscr{L}$ over X_U. In other words, the natural map $f^*(f_*\mathscr{L}) \to L$ is *surjective*, and hence (by the universal property of projective space) defines an S-morphism

$$\iota : X \to \mathrm{Proj}_S(f_*\mathscr{L}).$$

If we can show that ι is a closed immersion, we will be done.

The construction of ι commutes with any base change on S (check!), and on geometric fibers it recovers exactly the anti-canonical embedding of a genus-0 curve as a conic in the plane when $g = 0$ and the tricanonical embedding of a curve of genus g when $g > 1$. Thus, the induced map ι_s on s-fibers is a closed immersion for all $s \in S$. It follows that the proper S-map ι is quasi-finite, hence *finite*. The closed immersion property on s-fibers therefore enables us to infer via Nakayama's Lemma that ι is a closed immersion (why?). ■

4. Rigidity

We conclude with a result which is an answer to many prayers for converting equalities of maps on fibers into equality of maps over a base. This will not involve the formalism of base change for cohomology, except that it uses a condition whose only natural means of verification is via that formalism. The basic setup is as follows.

Consider a pair of S-maps $f_1, f_2 : X \rightrightarrows Y$ between two S-schemes, where $p : X \to S$ is proper and $\mathscr{O}_S = p_*\mathscr{O}_X$. (By Corollary 1.3, we have criteria to verify this final condition!) Loosely speaking, $X \to S$ is akin to a family of compact connected complex manifolds. If $(f_1)_s = (f_2)_s$ for all $s \in S$, when can we conclude that $f_1 = f_2$ as scheme morphisms? We stress that this is a nontrivial problem when S is an artin local ring, and the question is only interesting when X is non-reduced (as otherwise we can replace Y with Y_{red} and then everything is obvious since by assumption f_1 and f_2 coincide topologically). A natural context for this kind of question is when relativizing results such as "every map between abelian varieties that respect the identity sections is a homomorphism" to the theory over a base scheme.

Let's focus on the case that $Y = G$ is an S-group. In this case we can form the map $f_1 \cdot (f_2)^{-1}$ and thereby reduce the problem that of asking whether $f : X \to G$ is the constant map through the identity section (i.e., $f = e_G \circ p$) if it is so on fibers. Certainly some additional hypothesis is needed, as otherwise we could take $f = g \circ p$ for a nontrivial section $g \in G(S)$ that is topologically supported at the identity. (For example, if $S = \mathrm{Spec}(k[\epsilon])$ then g could correspond to a nonzero tangent vector at the identity.) So the "best possible" result would be that $f = g \circ p$ for some $g \in G(S)$. If that were the case, and *if* there exists some $x \in X(S)$ such that $f \circ x = e$ in $G(S)$ then we would win: $e = f \circ x = g \circ p \circ x = g \circ 1_S = g$. In other words, if f induces the identity map on fibers and carries some $x \in X(S)$ over to $e \in G(S)$ then we would get $f = e \circ p$ as desired. Thus, the following result is what we want:

Proposition 4.1 (Rigidity)**.** *Let S be a scheme and $f : X \to Y$ an S-map where $p : X \to S$ is closed and $\mathscr{O}_S \simeq p_*\mathscr{O}_X$. If for every geometric point $\overline{s}$ of S the fiber map $f_{\overline{s}} : X_{\overline{s}} \to Y_{\overline{s}}$ factors through a $k(\overline{s})$-point of $Y_{\overline{s}}$ then $f = y \circ p$ for a unique $y \in Y(S)$.*

Proof. First we check uniqueness. Since $p(X) \subseteq S$ is a closed set, p must be surjective because restricting over $S - p(X)$ would bring us to the case when X is empty and $\mathscr{O}_S = p_*\mathscr{O}_X$, forcing S to be empty. Thus, the condition $y \circ p = f$ determines y topologically. To prove that the map $y^\sharp : \mathscr{O}_Y \to y_*\mathscr{O}_S$ is uniquely determined, we note that inserting the isomorphism $p^\sharp : \mathscr{O}_S \simeq p_*\mathscr{O}_X$ converts the unknown $y^\sharp$ into the map

$\mathscr{O}_Y \to f_*\mathscr{O}_X$ that is exactly $(y \circ p)^\sharp = f^\sharp$ (check!). Hence, y is uniquely determined as a map of ringed spaces if it exists.

Now we run the argument in reverse to prove existence. By the (geometric) fibral hypothesis, f factors set-theoretically through p (i.e., $f(x)$ depends only on $p(x) \in S$). But p is a surjective closed map, hence a topological quotient map, so the resulting set-theoretic factorization $y : S \to Y$ satisfying $y \circ p = f$ (topologically) has y actually continuous. To promote y to a morphism of ringed spaces, we define $y^\sharp$ to be the composite map

$$\mathscr{O}_Y \to f_*\mathscr{O}_X = y_*(p_*\mathscr{O}_X) \simeq y_*\mathscr{O}_S$$

where the first step is $f^\sharp$ and the last step is the inverse to $y_*(p^\sharp)$. Now $f = y \circ p$ as maps of ringed spaces, and it remains to check that y is *locally ringed* (so y is really a morphism of schemes). That is, for $s \in S$ we want $\mathscr{O}_{Y,y(s)} \to \mathscr{O}_{S,s}$ to be a local map. Choosing $x \in p^{-1}(s)$ (so $f(x) = y(p(x)) = y(s)$), composing with the local map $\mathscr{O}_{S,s} \to \mathscr{O}_{X,x}$ induced by $p^\sharp$ yields the local map $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ induced by $f^\sharp$ (check!). But it is easy to check that a ring homomorphism $R \to R'$ between local rings is a local map provided that its composition with a local map $R' \to R''$ is local. ■